2 The Derivative

2.2 Differentiability in One Variable

We seek to view the derivative in a way that generalizes to multiple dimensions. From our studies of functions $f : \mathbb{R} \to \mathbb{R}$ and $f : \mathbb{R}^2 \to \mathbb{R}$, we have the idea of a tangent line/plane approximation. That is, as we get closer and closer to the point of tangency, our tangent line (resp. plane) and curve (resp. surface) get closer and closer. We derive a more mathematically precise way to describe this.

Let $f : \mathbb{R} \to \mathbb{R}$ be a function, and say a is the point at which we want to construct our tangent line $l(x) = mx + b$. Our first obvious constraint on l is that it must intersect f at the point of tangency, meaning $l(a) = f(a)$. This forces $ma + b = f(a)$, and so $b = f(a) - ma$. Plugging this back into our equation for the line, we get $l(x) = f(a) + m(x - a)$. Thus, the family of all potential tangent lines reduces from all lines to only the lines passing through $(a, f(a))$. Great! But how do we pin down the correct slope m?

[2.2.1] Example

To see what goes wrong with a bad guess, let's isolate a specific example: $f(x) = x^2$ at $a = 0$. Since $f(0) = 0$, our candidate tangent lines take the form $l(x) = mx$. So...is $l(x) = 5x$ a tangent line? Obviously not—we want our tangent line to fit snugly with f near $x = 0$.

Try slopes like $2x$, x, or $0.0001x$. At some scale, these lines might *look* tangent to f at $x = 0$, but if we zoom in close enough, they all inevitably stab through the parabola rather than hugging it. While the absolute gap between the curve and the line vanishes as $x \to 0$, it shrinks at the *same rate* that x itself is shrinking.

Returning to our general $f(x)$ and $l(x)$, we see that merely requiring $l(x) \to f(x)$ as $x \to a$ isn't enough; *any* intersecting line satisfies that! For $l(x)$ to fit snugly with $f(x)$ as the true tangent line, we want $l(x)$ to get closer to $f(x)$ at a rate *faster* than $x \to a$. If the gap between $f(x)$ and $l(x)$ shrinks at the same rate as $x \to a$, then the relative error between them remains constant, meaning the line intersects the curve at a sharp angle. To fit our notion of "tangency," the error must be a smaller order of magnitude than the distance we are zooming in.

[2.2.2] Definition (*Tangent Line*)

This behavior is perfectly captured by little-o notation. We demand that the error $f(x) - l(x)$ is $o(|x - a|)$. In other words, as $x \to a$, the ratio of the error to the distance $|x - a|$ must completely vanish:

$$\lim_{x \to a} \frac{f(x) - l(x)}{|x - a|} = 0.$$

[2.2.3] Theorem (*Equivalence of Little-o and Limit Definitions*)

Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function, and let $l(x) = f(a) + m(x - a)$ be a generic line passing through $(a, f(a))$. The error $f(x) - l(x)$ is $o(|x - a|)$ if and only if $m = f'(a)$.

Proof. By definition, the error being $o(|x-a|)$ implies that

$$\lim_{x\to a} \frac{f(x)-(f(a)+m(x-a))}{x-a} = 0.$$

Splitting the fraction, we get

$$\lim_{x\to a} \left(\frac{f(x)-f(a)}{x-a} - \frac{m(x-a)}{x-a}\right) = 0,$$

which evaluates to

$$\lim_{x\to a} \left(\frac{f(x)-f(a)}{x-a}\right) - m = 0.$$

Rearranging to solve for m yields

$$m = \lim_{x\to a} \frac{f(x)-f(a)}{x-a},$$

which is exactly $f'(a)$. By simply enforcing that a generic line intersects $f(a)$ and that its error is $o(|x-a|)$, the only valid slope remaining is exactly the standard limit definition of the derivative. ❦