1 Preliminaries to Complex Analysis

1.1 Complex Numbers and the Complex Plane

[1.1.1] Definition (*Complex Numbers*)

A **complex number** is an expression of the form $z = x + iy$, where $x, y \in \mathbb{R}$ and $i^2 = -1$.

(1) We write $x = \mathrm{Re}(z)$ for the **real part** of z, and $y = \mathrm{Im}(z)$ for the **imaginary part** of z.

(2) If $\mathrm{Re}(z) = 0$, then z is **purely imaginary**.

(3) The set of all complex numbers is denoted by $\mathbb{C}$.

[1.1.2] Remark (*Vector Space Isomorphism*)

$\mathbb{C}$ is isomorphic to the Euclidean plane $\mathbb{R}^2$ as a real vector space via the canonical bijection:

$$z = x + iy \longleftrightarrow (x, y) \in \mathbb{R}^2$$

Under this identification, the real numbers correspond to the x-axis (the real axis) and the purely imaginary numbers to the y-axis (the imaginary axis).

[1.1.3] Definition (*Absolute Value and Conjugation*)

Let $z = x + iy \in \mathbb{C}$.

(1) The **absolute value** (or modulus) of z is the Euclidean length of its corresponding vector in $\mathbb{R}^2$:

$$|z| = (x^2 + y^2)^{1/2}$$

(2) The **complex conjugate** of z is its reflection across the real axis:

$$\overline{z} = x - iy$$

[1.1.4] Proposition (*Basic Arithmetic and Metric Properties*)

Let $z, z_1, z_2 \in \mathbb{C}$. The following properties hold:

(1) $\mathrm{Re}(z) = \frac{z+\overline{z}}{2}$ and $\mathrm{Im}(z) = \frac{z-\overline{z}}{2i}$.

(2) $|z|^2 = z\overline{z}$.

(3) If $z \neq 0$, then $\frac{1}{z} = \frac{\overline{z}}{|z|^2}$.

(4) **Triangle Inequality**: $|z_1 + z_2| \le |z_1| + |z_2|$.

(5) **Reverse Triangle Inequality**: $||z_1| - |z_2|| \le |z_1 - z_2|$.

[1.1.5] Definition (*Polar Form*)

Any non-zero complex number $z \in \mathbb{C}$ can be written in **polar form**:

$$z = re^{i\theta}$$

where $r = |z| > 0$ is the modulus, and $\theta \in \mathbb{R}$ is the **argument** of z (defined uniquely up to a multiple of 2π), denoted by $\arg z$. By Euler's formula:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

[1.1.6] Remark (*Geometric Multiplication*)

If $z = re^{i\theta}$ and $w = se^{i\varphi}$, then $zw = rse^{i(\theta+\varphi)}$. Thus, multiplication by a complex number corresponds geometrically to a homothety in $\mathbb{R}^2$ (a rotation by $\arg z$ composed with a dilation by $|z|$).

1.2 Convergence

[1.2.1] Definition (*Convergence of Sequences*)

A sequence $\{z_n\} \subset \mathbb{C}$ **converges** to $w \in \mathbb{C}$ if:

$$\lim_{n\to\infty} |z_n - w| = 0$$

We write $w = \lim_{n\to\infty} z_n$.

[1.2.2] Proposition (*Component-wise Convergence*)

A sequence $\{z_n\} \subset \mathbb{C}$ converges to w if and only if the sequences of real and imaginary parts of z_n converge to the real and imaginary parts of w, respectively.

[1.2.3] Definition (*Cauchy Sequence*)

A sequence $\{z_n\} \subset \mathbb{C}$ is a **Cauchy sequence** if for every $\varepsilon > 0$, there exists an integer $N > 0$ such that:

$$|z_n - z_m| < \varepsilon \quad \text{whenever } n, m > N$$

[1.2.4] Theorem (*Completeness of* $\mathbb{C}$)

The complex numbers $\mathbb{C}$ form a complete metric space. That is, every Cauchy sequence in $\mathbb{C}$ converges to a limit in $\mathbb{C}$.

> **Proof.** Since $\{z_n\}$ is Cauchy if and only if the component sequences $\{\mathrm{Re}(z_n)\}$ and $\{\mathrm{Im}(z_n)\}$ are Cauchy in $\mathbb{R}$, completeness of $\mathbb{C}$ follows directly from the completeness of $\mathbb{R}$. ❦

1.3 Sets in the Complex Plane

[1.3.1] Definition (*Discs*)

Let $z_0 \in \mathbb{C}$ and $r > 0$.

(1) The **open disc** of radius r centered at z_0 is:

$$D_r(z_0) = \{z \in \mathbb{C} : |z - z_0| < r\}$$

(2) The **closed disc** of radius r centered at z_0 is:

$$\overline{D}_r(z_0) = \{z \in \mathbb{C} : |z - z_0| \leq r\}$$

(3) The boundary of either diskis the circle $C_r(z_0) = \{z \in \mathbb{C} : |z - z_0| = r\}$.

(4) The **unit disc** is denoted by $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

[1.3.2] Definition (*Topological Vocabulary*)

Let $\Omega \subset \mathbb{C}$ be a set.

(1) A point $z_0 \in \Omega$ is an **interior point** of Ω if there exists $r > 0$ such that $D_r(z_0) \subset \Omega$.

(2) The **interior** of Ω, denoted Ω°, is the set of all its interior points.

(3) Ω is **open** if every point in Ω is an interior point (i.e., $\Omega = \Omega^\circ$).

(4) Ω is **closed** if its complement $\Omega^c = \mathbb{C} \setminus \Omega$ is open.

(5) $z \in \mathbb{C}$ is a **limit point** of Ω if there is a sequence $\{z_n\} \subset \Omega$ with $z_n \neq z$ such that $z_n \to z$. A set is closed iff it contains all its limit points.

(6) The **closure** of Ω is $\overline{\Omega} = \Omega \cup \{\text{limit points}\}$.

(7) The **boundary** of Ω is $\partial\Omega = \overline{\Omega} \setminus \Omega^\circ$.

(8) Ω is **bounded** if there exists $M > 0$ such that $|z| < M$ for all $z \in \Omega$.

(9) If Ω is bounded, its **diameter** is:

$$\operatorname{diam}(\Omega) = \sup_{z,w \in \Omega} |z - w|$$

[1.3.3] Definition (*Compactness*)

A set $\Omega \subset \mathbb{C}$ is **compact** if it is closed and bounded.

[1.3.4] Theorem (*Characterizations of Compactness*)

Let $\Omega \subset \mathbb{C}$ be a subset. The following are equivalent:

(1) Ω is compact (closed and bounded).

(2) Every sequence $\{z_n\} \subset \Omega$ has a subsequence that converges to a point in Ω (sequential compactness).

(3) Every open covering of Ω admits a finite subcovering.

[1.3.5] Proposition (*Nested Compact Sets*)

If $\Omega_1 \supset \Omega_2 \supset \cdots \supset \Omega_n \supset \cdots$ is a nested sequence of non-empty compact sets in $\mathbb{C}$ with $\operatorname{diam}(\Omega_n) \to 0$ as $n \to \infty$, then there exists a unique point $w \in \mathbb{C}$ such that $w \in \Omega_n$ for all n.

[1.3.6] Definition (*Connectedness and Regions*)

An open set $\Omega \subset \mathbb{C}$ is **connected** if it is not possible to find two disjoint, non-empty open sets Ω_1, Ω_2 such that $\Omega = \Omega_1 \cup \Omega_2$.

(1) A connected open set in $\mathbb{C}$ is called a **region**.

(2) For open sets in $\mathbb{C}$, connectedness is equivalent to **path-connectedness**: any two points in Ω can be joined by a continuous curve entirely contained in Ω.

1.4 Continuous Functions

[1.4.1] Definition (*Continuity*)

Let f be a complex-valued function defined on $\Omega \subset \mathbb{C}$.

(1) f is **continuous** at $z_0 \in \Omega$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that:

$$|f(z) - f(z_0)| < \varepsilon \quad \text{whenever } z \in \Omega \text{ and } |z - z_0| < \delta$$

(2) Equivalently, f is continuous at z_0 if for every sequence $z_n \to z_0$ in Ω, we have $f(z_n) \to f(z_0)$.

[1.4.2] Theorem (*Extreme Value Theorem*)

A continuous function f on a compact set $\Omega \subset \mathbb{C}$ is bounded and attains both a maximum and a minimum value on Ω (where maximum/minimum are defined in terms of the real-valued modulus $|f(z)|$).

Note that there is no analogue of the intermediate value theorem, since $\mathbb{C}$ does not admit a standard ordering that respects its structure.

1.5 Holomorphic Functions

[1.5.1] Definition (*Holomorphicity*)

Suppose $\Omega \subseteq \mathbb{C}$ is an open set with $z_0 \in \Omega$. Then a function f is said to be **holomorphic at** z_0 if the value

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} \qquad h \in \mathbb{C}$$

exists.

A function f is said to be **holomorphic** on Ω if it is holomorphic at every point in Ω. For a general, non-open set $C \subseteq \mathbb{C}$, we say f is holomorphic on C if it is holomorphic on some open set containing C.

Note that every point in an open set is an interior point, and so the open disk$D_r(z_0) \subseteq \Omega$ for some small $r > 0$ (ensuring $z_0 + h \in \Omega$ for all complex h with $|h| < r$, which prevents vacuous edge-cases). This is the first definition that is truly *complex* in nature. While this resembles the real definition of differentiability, note that h is not real-valued, but rather $h \in (\mathbb{C} \cong \mathbb{R}^2)$. We have made complex differentiability far more strict: this limit must exist not just when approaching from the left and right as in $\mathbb{R}$, but along any path, from every available angle, as $h \to 0$ in the complex plane. Of course, also note that the limit that defines $f'(z_0)$ is referred to as the (complex) derivative of f at z_0.

[1.5.2] Theorem (*Holomorphicity Implies Continuity*)

Suppose f is holomorphic at some z_0 in f's domain. Then f is continuous at z_0.

Proof. We show $f(z_0 + h) \to f(z_0)$ as $h \to 0$ as such:

$$\lim_{h \to 0} f(z_0 + h) - f(z_0) = \lim_{h \to 0} \left(h \times \frac{f(z_0 + h) - f(z_0)}{h} \right) = 0 \times f'(z_0) = 0.$$

❦

Rearranging our definition of a complex derivative, we have that

$$\lim_{h\to 0}\frac{f(z_0+h)-f(z_0)}{h}=a \implies \underbrace{[f(z_0+h)-f(z_0)]}_{\Delta f}-ah=o(h),$$

where $o(h)$ is some map such that $\lim_{h\to 0} o(h)/h=0$. This statement says that, for $h\in\mathbb{C}$ of small magnitude, we can approximate the deviation of f from z_0 (denoted $\Delta f=f(z_0+h)-f(z_0)$) very well with the linear change ah. This approximation is so good that the error goes to zero much quicker than h can go to 0. Note that limits (and thus derivatives) are unique, and so a is the unique constant that approximates Δf by the linear function ah. In multivariate theory, we say that the linear map $h\mapsto ah$ is the true derivative of f, and this idea can be extended to all forms of finite-dimensional domains and codomains.

[1.5.3] Proposition (*Basic Differentiation Rules for* $\mathbb{C}$)

Suppose f,g are holomorphic functions on Ω.

(1) $f+g$ is holomorphic in Ω, with $(f+g)'=f'+g'$.

(2) fg is holomorphic in Ω, with $(fg)'=f'g+fg'$.

(3) f/g is holomorphic in Ω at points for which $g\neq 0$, with $(f/g)'=(f'g-fg')/g^2$.

(4) The chain rule holds. For the two-function case $f:\Omega\to U$ and $g:U\to\mathbb{C}$, where both are holomorphic, we have that

$$(g\circ f)'=g'(f(z))\,f'(z)\quad \forall z\in\Omega.$$

From this, we can prove basic results that polynomials, rational functions, and other nice classes of functions are holomorphic for suitable domains.

1.6 Complex-Valued Functions as Mappings

[1.6.1] Remark (*Real Differentiability vs. Complex Differentiability*)

We now make the distinction between treating a complex-valued function as a function $\mathbb{R}^2\to\mathbb{R}^2$. Consider the function $f(z)=\overline{z}$, then note

$$f'(z)=\lim_{h\to 0}\frac{f(z+h)-f(z)}{h}=\lim_{h\to 0}\frac{\overline{h}}{h},$$

which does not exist (parameterizing $\overline{z}/z$ by an angle yields $\exp(-2i\theta)$, and so the limit varies based on path chosen). Thus $f:\mathbb{C}\to\mathbb{C}$ is not holomorphic. But now consider $g:\mathbb{R}^2\to\mathbb{R}^2$ defined by $g(x,y)=(u(x,y),v(x,y))=(x,-y)$. This function, as a real-valued one, is linear, infinitely differentiable, and enjoys many properties. One such property is that its total derivative (as a linear map), expressed in the standard basis of $\mathbb{R}^2$, is written as

$$g'=\begin{bmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}.$$

From the perspective of multivariate theory, a map $g:\mathbb{R}^2\to\mathbb{R}^2$ has a derivative that is a linear map in two-dimensions, in contrast with a map $f:\mathbb{C}\to\mathbb{C}$ having a singular, complex number for its derivative (that is also a one-dimensional linear map).

[1.6.2] Remark (*Cauchy-Riemann Equations*)

We can link real differentiability and complex differentiability, however. For a complex function $f:\Omega\to\mathbb{C}$ to be differentiable, its difference function must take on the same value as $h\to 0$ through any path. Writing $h=h_1+ih_2$ and $z=x+iy$ (and so $f(z)=f(x,y)$), we may choose a point $z_0=x_0+iy_0\in\Omega$ for which $f'(z_0)$ exists and yield the following:

$$\begin{aligned} f'(z_0)=\lim_{h\to 0}\frac{f(z_0+h)-f(z_0)}{h} &= \lim_{h_1\to 0}\frac{f(x_0+h_1,y_0)-f(x_0,y_0)}{h_1}=\frac{\partial f}{\partial x}(x_0) \quad (h_2=0)\\ &= \lim_{h_2\to 0}\frac{f(x_0,y_0+h_2)-f(x_0,y_0)}{ih_2}=\frac{1}{i}\frac{\partial f}{\partial y}(x_0) \quad (h_1=0). \end{aligned}$$

If f is holomorphic, we have that $\frac{\partial f}{\partial x}=\frac{1}{i}\frac{\partial f}{\partial y}=-i\frac{\partial f}{\partial y}$ by the uniqueness of the derivative. If we write $f(z)=u(z)+iv(z)$, then we have that the partials u_x,u_y,v_x,v_y all exist, and they are such that

$$\frac{\partial f}{\partial x}=-i\frac{\partial f}{\partial y}\implies\left(\frac{\partial u}{\partial x}\right)+i\left(\frac{\partial v}{\partial x}\right)=i\left[\left(\frac{\partial u}{\partial y}\right)+i\left(\frac{\partial v}{\partial y}\right)\right]=\left(-\frac{\partial v}{\partial y}\right)+i\left(\frac{\partial u}{\partial y}\right),$$

implying the **Cauchy-Riemann equations**:

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \quad \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}.$$

[1.6.3] Remark (*Derivation of Wirtinger Operators*)

We will motivate the introduction of two important differential operators known as Wirtinger operators. To get started, let $z\in\mathbb{C}$ be a complex variable. Noting that $\mathbb{C}\cong\mathbb{R}^2$, write $z=x+iy$ and $\overline{z}=x-iy$. We may "solve" for x,y in terms of $z,\overline{z}$ as such:

$$x=\frac{z+\overline{z}}{2} \quad y=\frac{z-\overline{z}}{2i}.$$

Treating $\overline{z}$ as independent of z, we can expand the differential operators $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \overline{z}}$ by the multivariate chain rule.

$$\frac{\partial}{\partial z}=\underbrace{\frac{\partial x}{\partial z}}_{1/2}\frac{\partial}{\partial x}+\underbrace{\frac{\partial y}{\partial z}}_{1/2i}\frac{\partial}{\partial y}=\frac{1}{2}\left(\frac{\partial}{\partial x}+\frac{1}{i}\frac{\partial}{\partial y}\right),$$

$$\frac{\partial}{\partial \overline{z}}=\underbrace{\frac{\partial x}{\partial \overline{z}}}_{1/2}\frac{\partial}{\partial x}+\underbrace{\frac{\partial y}{\partial \overline{z}}}_{-1/2i}\frac{\partial}{\partial y}=\frac{1}{2}\left(\frac{\partial}{\partial x}-\frac{1}{1}\frac{\partial}{\partial y}\right).$$

[1.6.4] Proposition

If $f(z) = u(z) + iv(z)$ is holomorphic at z_0, then

$$\frac{\partial f}{\partial \bar{z}}(z_0) = 0 \quad f'(z_0) = \frac{\partial f}{\partial z}(z_0) = 2\frac{\partial u}{\partial z}(z_0).$$

Let F be a mapping in terms of real variables such that $F(x, y) = f(x + iy)$. Then we have that F is differentiable, and its derivative can be represented by a Jacobian J_F such that

$$\det\big(J_F(x_0, y_0)\big) = |f(x_0 + iy_0)|^2.$$

Proof. Suppose f is holomorphic at z_0. Then f satisfies the Cauchy-Riemann equations, meaning $\frac{\partial f}{\partial x}(z_0) = \frac{1}{i}\frac{\partial f}{\partial y}$, and so

$$\frac{\partial f}{\partial \bar{z}}(z_0) = \frac{1}{2}\left(\frac{\partial f}{\partial x}(z_0) - \frac{1}{i}\frac{\partial f}{\partial y}(z_0)\right) = 0.$$

From this, we have that

$$f'(z_0) = \frac{1}{2}\left(\frac{\partial f}{\partial x}(z_0) + \frac{\partial f}{\partial x}(z_0)\right) = \frac{1}{2}\left(\frac{\partial f}{\partial x}(z_0) + \frac{1}{i}\frac{\partial f}{\partial y}(z_0)\right) = \frac{\partial f}{\partial z}(z_0).$$

From the Cauchy-Riemann Equations, we get that $\frac{\partial f}{\partial z} = 2\frac{\partial u}{\partial z}$ for $f = u + iv$. We have already proved that complex differentiability implies real differentiability by the Cauchy-Riemann equations. Finally, note that

$$J_F(x_0, y_0) = \begin{bmatrix} u_x(x_0) & u_y(y_0) \\ v_x(x_0) & v_y(y_0) \end{bmatrix},$$

and so $\det\big(J_F(x_0, y_0)\big) = u_x(x_0)v_y(y_0) - u_y(y_0)v_x(x_0) = (u_x(x_0))^2 + (u_y(y_0))^2 = |f'(z_0)|^2$.

[1.6.5] Theorem (*Criterion for Holomorphicity*)

A function $f(z) = u(z) + iv(z)$ is **holomorphic** if and only if u, v have continuous, first-order partial derivatives that satisfy the Cauchy-Riemann equations.

Proof.

$(\Longrightarrow)$: This is implied from taking the limit of the difference quotient along $\mathbb{R}$ and $i\mathbb{R}$.

$(\Longleftarrow)$: Suppose u, v have continuous first-order partial derivatives that satisfy the Cauchy-Riemann equations. From multivariate theory, note that

$$u(x+h, y+k) - u(x, y) = u_x h + u_y k + o_1(h + ik),$$
$$v(x+h, y+k) - v(x, y) = v_x h + v_y k + o_2(h + ik).$$

Then note

$$f(z + [h + ik]) - f(z) = (u_x + iv_x)(h + ik) + [o_1(h + ik) + io_2(h + ik)],$$

meaning

$$f'(z) = \lim_{h+ik\to 0} \frac{f(z + [h + ik]) - f(z)}{h + ik} = u_x + iv_x.$$

1.7 Power Series

[1.7.1] Definition (*Complex Exponential and Trigonometric Functions*)

The complex **exponential function** is defined for all $z \in \mathbb{C}$ by the power series:

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

This series converges absolutely for every $z \in \mathbb{C}$, and converges uniformly in every closed diskin $\mathbb{C}$.

Using the complex exponential, the standard complex **trigonometric functions** are defined by:

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} = \frac{e^{iz} + e^{-iz}}{2},$$

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} = \frac{e^{iz} - e^{-iz}}{2i}.$$

These definitions yield the **Euler formulas** and agree with their real counterparts for $z \in \mathbb{R}$.

[1.7.2] Theorem (*Hadamard's Formula and Convergence*)

Given a power series $\sum_{n=0}^{\infty} a_n z^n$, there exists a **radius of convergence** R ($0 \le R \le \infty$) such that:

(1) If $|z| < R$, the series converges absolutely.

(2) If $|z| > R$, the series diverges.

Under the conventions $1/0 = \infty$ and $1/\infty = 0$, R is determined by **Hadamard's formula**:

$$1/R = \limsup_{n\to\infty} |a_n|^{1/n}.$$

The open region $D_R(0) = \{z \in \mathbb{C} : |z| < R\}$ is called the **disk of convergence**.

Proof. Let $L = 1/R = \limsup_{n\to\infty} |a_n|^{1/n}$.

If $|z| < R$, then $L|z| < 1$. Choose $\varepsilon > 0$ sufficiently small such that $(L+\varepsilon)|z| = r < 1$. By the definition of the limit superior, we have $|a_n|^{1/n} \le L + \varepsilon$ for all sufficiently large n. Thus:

$$|a_n||z|^n \le [(L+\varepsilon)|z|]^n = r^n.$$

Since $r < 1$, absolute convergence follows by comparison with the convergent geometric series $\sum r^n$.

If $|z| > R$, then $L|z| > 1$. A similar argument shows that $|a_n||z|^n > 1$ infinitely often, so the terms do not tend to 0, forcing divergence. ❦

[1.7.3] Remark

The behavior of a power series on its boundary $|z| = R$ is delicate; one can have either convergence or divergence depending on the specific coefficients (e.g., $\sum z^n$, $\sum z^n/n^2$, and $\sum z^n/n$ on the unit circle $\partial\mathbb{D}$).

[1.7.4] THEOREM (*Holomorphicity of Power Series*)

The power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ defines a holomorphic function in its disk of convergence. Furthermore, $f'(z)$ can be obtained by term-by-term differentiation:

$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1},$$

and this differentiated series has the same radius of convergence R.

[1.7.5] COROLLARY (*Infinite Complex Differentiability*)

A power series $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ is infinitely complex differentiable in its disk of convergence, and all higher-order derivatives are obtained by term-by-term differentiation.

[1.7.6] DEFINITION (*Analytic Functions*)

A complex-valued function f defined on an open set $\Omega \subseteq \mathbb{C}$ is said to be **analytic** (or to have a **power series expansion**) at $point$ $z_0 \in \Omega$ if there exists a power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ with a positive radius of convergence such that:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad \text{for all } z \text{ in a neighborhood of } z_0.$$

If f is analytic at every point in Ω, we say f is **analytic on** Ω.

[1.7.7] REMARK

By Theorem 2.6, any analytic function on Ω is automatically holomorphic on Ω. A fundamental and deep result in complex analysis (the converse, which we prove in Chapter 2) is that every holomorphic function is analytic. Thus, we will use the terms *holomorphic* and *analytic* interchangeably.

1.8 Integration along Curves

[1.8.1] Definition (*Parameterized Curves, Smooth Curves, Piecewise-Smooth Curves*)

A **parameterized curve** is some function $z(t)$ that maps $[a,b] \subseteq \mathbb{R}$ into the complex plane $\mathbb{C}$. We say that the parameterized curve is **smooth** if $z'(t)$ exists, is continuous, and is never equal to 0 for every $t \in [a,b]$. We say that the parameterized curve is **piecewise-smooth** if z is continuous on $[a,b]$ and has a partition

$$a = a_0 < a_1 < \cdots < a_{n-1} < a_n = b$$

such that $z(t)$ is smooth when restricted to any subinterval $[a_k, a_{k+1}]$. Note that this definition does not require the left and right-hand derivatives for z' to agree at each a_k for $k \in \{1, \ldots, n-1\}$.

For any circle of radius r centered at z_0 in the complex plane, we note that $z(t) = z_0 + re^{it}$ for $t \in [0, 2\pi]$ is a smooth parameterization of it. But note that $z_0 + re^{2it}$ parameterizes the same curve for the same domain, and so does $z_0 + re^{4it}$, and infinitely more parameterizations. We seek a notion of equality between these curves.

[1.8.2] Definition (*Equivalence of Parameterized Curves*)

Two parameterizations $z : [a,b] \to \mathbb{C}$ and $\tilde{z} : [c,d] \to \mathbb{C}$ are **equivalent** if there exists a continuously differentiable map $[c,d] \to [a,b]$ defined by $s \mapsto t(s)$ such that $\tilde{z}(s) = z(t(s))$ and $t'(s) > 0$ for every $s \in [c,d]$.

Note the condition that $t'(s) > 0$ in the previous definition ensures that as s ticks forward, $t(s)$ also ticks forward, meaning z can never change direction. Thus equivalent curves share the same direction/orientation, and so a curve and its reverse aren't equivalent curves.

Equality in this sense is an equivalence relation, and so equivalent curves form equivalence classes. When referring to a curve γ with no explicit parameterization, we refer to the entire equivalence class of functions that can parameterize γ. That is, a curve represents the set of all valid paths tracing the exact same geometric set of points with the same orientation (that is, γ and $-\gamma$ are not necessarily equal).

[1.8.3] Definition (*Closed and Simple Curves*)

A smooth curve parameterized by $z : [a,b] \to \mathbb{C}$ is called **closed** if $z(a) = z(b)$. A curve is called **simple** if $z(s) \neq z(t)$ for any $s \neq t$ (i.e., the curve never self-intersects). A closed, simple curve is a closed curve that is simple when disregarding its endpoints.

[1.8.4] Definition (*Integral Along Smooth and Piecewise-Smooth Curve*)

Suppose γ is a smooth curve, and let f be a continuous function on γ. Let $z:[a,b]\to\mathbb{C}$ be any parameterization of γ, and define the **integral of f along** γ by

$$\int_\gamma f(z)\,\mathrm{d}z = \int_a^b f(z(t))\,z'(t)\,\mathrm{d}t.$$

If γ is piecewise-smooth, let $z:[a,b]\to\mathbb{C}$ be any piecewise-smooth parameterization over the partition $a=a_0<a_1<\cdots<a_n=b$. Then the integral of f along γ is defined by

$$\int_\gamma f(z)\,\mathrm{d}z = \sum_{k=0}^{n-1}\int_{a_k}^{a_{k+1}} f(z(t))\,z'(t)\,\mathrm{d}t.$$

[1.8.5] Theorem (*Integral Along Curve Does Not Depend on Parameterization*)

Suppose γ is a smooth curve. Then the integral of f along γ is independent of parameterization.

Proof. Let $z:[a,b]\to\mathbb{C}$ and $\tilde{z}:[c,d]\to\mathbb{C}$ be equivalent parameterizations of γ. Thus there is some continuously bijective map $s\mapsto t(s)$ such that $t'(s)>0$ and $\tilde{z}(s)=z(t(s))$ for every $s\in[c,d]$. Then note

$$\begin{aligned}
\int_\gamma f(z)\,\mathrm{d}z &= \int_a^b f(z(t))\,z'(t)\,\mathrm{d}t \\
&= \int_c^d f(z(t(s)))\,z'(t(s))\;t'(s)\,\mathrm{d}s && \text{[Change of Variables]} \\
&= \int_c^d f(\tilde{z}(s))\,\tilde{z}(s)\,\mathrm{d}s && \text{[Multivariate Chain Rule]} \\
&= \int_\gamma f(z)\,\mathrm{d}z.
\end{aligned}$$

❦

[1.8.6] Definition (*Length of Curve*)

Let γ be a smooth curve with a parameterization $z:[a,b]\to\mathbb{C}$. Then the **length** of γ is given by

$$\text{length}(\gamma) = \int_\gamma \mathrm{d}s = \int_a^b |z'(t)|\,\mathrm{d}t.$$

[1.8.7] Proposition (*Properties of Contour Integrals*)

Suppose f, g are continuous functions and γ is a smooth curve.

(1) **Linearity.** $\displaystyle\int_\gamma (af(z)+bg(z))\,\mathrm{d}z = a\int_\gamma f(z)\,\mathrm{d}z + b\int_\gamma g(z)\,\mathrm{d}z.$

(2) **Path Reversal.** $\displaystyle\int_\gamma f(z)\,\mathrm{d}z = -\int_{-\gamma} f(z)\,\mathrm{d}z.$

(3) **ML/Estimation Lemma.** $\displaystyle\left|\int_\gamma f(z)\,\mathrm{d}z\right| \le \sup_{z\in\gamma}|f(z)| \times \mathrm{length}(\gamma).$

Proof.

(1) Follows from definition.

(2) Let $z_1 : [a,b] \to \mathbb{C}$ be a parameterization for γ, and let $z_2 : [b,a] \to \mathbb{C}$ be a parameterization for $-\gamma$ such that $z_2(t) = z_1(b+a-t)$. Then note $z_2'(t) = -z_1'(b+a-t)$. Thus we yield

$$\int_\gamma f(z)\,\mathrm{d}z = \int_a^b f(z_1(t))\,z_1'(t)\,\mathrm{d}t = \int_b^a f(z_1(b+a-t))\,z_1'(b+a-t)\,\mathrm{d}t = -\int_b^a f(z_2(t))\,z_2'(t)\,\mathrm{d}t = -\int_{-\gamma} f(z)\,\mathrm{d}z.$$

(3) Let $z : [a,b] \to \mathbb{C}$ be a parameterization of γ.

$$\left|\int_\gamma f(z)\,\mathrm{d}z\right| = \left|\int_a^b f(z(t))\,z'(t)\,\mathrm{d}t\right| = \int_a^b \big|f(z(t))\big|\big|z'(t)\big|\,\mathrm{d}t \le \sup_{t\in[a,b]} |f(z(t))| \int_a^b |z'(t)|\,\mathrm{d}t = \sup_{z\in\gamma}|f'(z)| \times \mathrm{length}(\gamma).$$

❦

[1.8.8] Definition (*Primitive*)

Suppose f is a function on Ω. Then a function F that is holomorphic on Ω is called a **primitive** (or an **antiderivative**) of f if $F'(z) = f(z)$ for every $z \in \Omega$.

[1.8.9] Theorem (*FToC Applied to Contour Integration*)

If f is a continuous function on Ω that has a primitive F, and $\gamma \subseteq \Omega$ is a path from w_1 to w_2, we have that

$$\int_\gamma f(z)\,\mathrm{d}z = F(w_2) - F(w_1).$$

Proof. Suppose γ is smooth. Then let $z : [a,b] \to \mathbb{C}$ be a smooth parameterization of γ, and note

$$\begin{aligned}\int_\gamma f(z)\,\mathrm{d}z &= \int_a^b f(z(t))\,z'(t)\,\mathrm{d}t \\ &= \int_a^b F'(z(t))\,z'(t)\,\mathrm{d}t \\ &= \int_a^b \frac{\mathrm{d}}{\mathrm{d}x}F(z(t))\mathrm{d}t \\ &= F(z(b)) - F(z(a)) = F(w_2) - F(w_1).\end{aligned}$$

For γ piecewise-smooth, the exact same process occurs, but instead of an integral we use a sum over each partition. ❦

[1.8.10] Corollary (*Integral About Closed Contour*)

If γ is a closed curve in Ω, and f is continuous and has a primitive in Ω, then

$$\int_\gamma f(z)\,\mathrm{d}z = 0.$$

Note that $f(z) = 1/z$ does not admit a primitive in the punctured complex plane $\mathbb{C}^* = \mathbb{C}\setminus\{0\}$. Indeed, if we use the unit circle parameterized by $z = e^{it}$ for $t \in [0, 2\pi]$, we have that

$$\int_C \frac{1}{z}\,\mathrm{d}z = \int_0^{2\pi} \frac{1}{e^{it}} \times ie^{it}\,\mathrm{d}t = 2\pi i \neq 0,$$

and so $1/z$ does not have a primitive by contrapositive argumentation.

[1.8.11] Corollary

If f is holomorphic in a region Ω and $f' = 0$, then f is constant.

> **Proof.** Fix a point $w_0 \in \Omega$. Since Ω is a connected subset of $\mathbb{C}$, it is path-connected, so let $\gamma \subseteq \Omega$ be any path that terminates at w_0. Note then that
>
> $$0 = \int_\gamma f'(z) = f(w_0) - f(w),$$
>
> and so $f(w_0) = f(w)$. Since w is arbitrary, we have that f is constant on Ω. ❦

1.8.1 Exercises

[1.8.12] Problem

(1) Let γ be any circle about the origin. Evaluate $\displaystyle\int_\gamma z^n$ for all integers n.

(2) Evaluate the integral for any circle that is not centered at 0.

(3) Show that if $|a| < r < |b|$, then
$$\int_\gamma \frac{1}{(z-a)(z-b)}\,\mathrm{d}z = \frac{2\pi i}{b-a},$$
where γ is the circle of radius r about the origin.

Proof.

(1) Let γ be any circle about the origin. For simplicity, suppose $r = 1$, and let $z(t) = e^{it}$ be a parameterization of γ over $[0, 2\pi]$. Then note
$$\int_\gamma z^n\,\mathrm{d}z = \int_0^{2\pi} (e^{int})(ie^{it})\,\mathrm{d}t = i\int_0^{2\pi} e^{it(n+1)}\,\mathrm{d}t = \frac{1}{n+1}\left[e^{it(n+1)}\right]_{t=0}^{t=2\pi} = \frac{e^{2\pi i(n+1)}-1}{n+1} = \frac{1^{n+1}-1}{n+1} = 0$$
for all $n \neq -1$. Note that if we let r be arbitrary, then the result would simply be the integral scaled by r^n, which is still 0. In the case of $n = -1$, we have that
$$\int_\gamma \frac{1}{z}\,\mathrm{d}z = \int_0^{2\pi} \frac{i(re^{it})}{re^{it}} = 2\pi i.$$

(2) Note that z^n is holomorphic on $\mathbb{C}^*$ for all $n \in \mathbb{Z}_+$. Let Ω be an open set such that $\gamma \subseteq \Omega$, and so $\displaystyle\int_\gamma z^n\,\mathrm{d}z = 0$ by holomorphicity (and thus continuity) of the integrand and by γ being a closed curve.

(3) Split the integrand by the method of partial fractions:
$$\int_\gamma \frac{1}{(z-a)(z-b)}\,\mathrm{d}z = \underbrace{\left(\frac{1}{a-b}\int_\gamma \frac{1}{z-a}\,\mathrm{d}z\right)}_{:=I_1} + \underbrace{\left(\frac{1}{b-a}\int_\gamma \frac{1}{z-b}\,\mathrm{d}z\right)}_{:=I_2}.$$

I_2 is easy to evaluate.

[1.8.13] Problem

Suppose f is continuous on a region Ω. Prove that if f has a primitive on Ω, then it is unique.

Proof. Suppose f admits two primitives F_1, F_2 on Ω. Fix $w_0 \in \Omega$, and let $w \in \Omega$ vary. Since Ω is a region, it is path-connected, so suppose γ is a path from w to w_0. Then note
$$\int_\gamma f(z)\,\mathrm{d}z = F_1(w_0) - F_1(w) = F_2(w_0) - F_2(w) \implies F_1(w) - F_2(w) = \underbrace{F_1(w_0) - F_2(w_0)}_{:=C}.$$

Thus $F_1 - F_2 \equiv C$, and so these functions are equivalent up to a constant.