2 Cauchy's Theorem and Its Applicatins

We have proven that if f is continuous and admits a primitive F on an open set Ω, then for any path $\gamma \subseteq \mathbb{C}$ that starts at w and ends at w_0, we have that

$$\int_\gamma f(z)\,\mathrm{d}z = F(w_0) - F(w).$$

From this, we immediately have that if γ is a closed path, then the integral evaluates to 0.

This mirrors the fundamental theorem of calculus, with the exception that this doesn't apply to all continuous functions—the condition that f admits a primitive is crucial. In real variables, a continuous function automatically admits a primitive/antiderivative, but this is not necessarily true in complex variables. For example, consider the function $1/z$ that is entire on $\mathbb{C}^*$. For the unit circle γ, note that

$$\int_\gamma \frac{1}{z}\,\mathrm{d}z = \int_0^{2\pi} \frac{ie^{it}}{e^{it}}\,\mathrm{d}t = 2\pi i \neq 0,$$

and so by contrapositive, $1/z$ has no primitive on $\mathbb{C}^*$ in spite of its holomorphicity.

Explicitly finding a primitive for a function is admittedly difficult. We alter our hypotheses to be more tractable, yielding **Cauchy's theorem**. It states that if f is holomorphic on an open set Ω, then for any path $\gamma \subseteq \Omega$ whose interior is also contained on Ω, we have that

$$\int_\gamma f(z)\,\mathrm{d}z = 0.$$

We prove this for toy contours (i.e., for contours where the "interior" is intuitively obvious) and defer the proof of the full statement for simply closed (i.e., Jordan) curves to the future. We start by proving Goursat's theorem, which proves this statement for the case where γ is a triangle.

[2.0.1] Theorem (*Goursat's Theorem*)

Suppose f is holomorphic on an open set Ω. Let $T \subseteq \Omega$ be some path that traces out a triangle whose interior also lies in Ω. Then

$$\int_T f(z)\,\mathrm{d}z = 0.$$

Proof. Let $T^{(0)}$ be the triangle in question, whose orientation is fixed and considered "positive," and let $d^{(0)}, p^{(0)}$ represent the diameter and perimeter of these triangles respectively. Consider the midpoints of each side of $T^{(0)}$ and connect each of them by a straight line, partitioning $T^{(0)}$ into four triangles $T_k^{(0)}$ for $k = 1,2,3,4$. We choose the direction of each triangle in the diagram shown in the next remark block.

Since the contours that aren't on the sides of $T^{(0)}$ cancel out, we know that

$$\int_{T^{(0)}} f(z)\,\mathrm{d}z = \sum_{k=1}^{4} \int_{T_k^{(0)}} f(z)\,\mathrm{d}z.$$

By the triangle inequality, there exists a $k \in \{1,2,3,4\}$ such that

$$\left|\int_{T^{(0)}} f(z)\,\mathrm{d}z\right| \le 4\left|\int_{T_k^{(0)}} f(z)\,\mathrm{d}z\right|.$$

Let $T^{(1)} = T_k^{(0)}$, and then let $d^{(1)}, p^{(1)}$ be the diameter and perimeter for $T^{(1)}$. By geometry, we note that $d^{(1)} = d^{(0)}/2$ and $p^{(1)} = p^{(0)}/2$. We generate a sequence of triangles

$$T^{(0)}, T^{(1)}, \ldots, T^{(n)}, \ldots$$

with the properties that

$$\left|\int_{T^{(0)}} f(z)\,\mathrm{d}z\right| \le 4^n\left|\int_{T^{(n)}} f(z)\,\mathrm{d}z\right|, \quad d^{(n)} = d^{(0)}/2^n, \quad p^{(n)} = p^{(0)}/2^n.$$

Let $\mathcal{T}^{(k)} = \overline{T^{(k)}}$, a closed and bounded (and thus compact) subspace of $\mathbb{C}$ in its metric topology. Thus

$$\mathcal{T}^{(0)} \supseteq \mathcal{T}^{(1)} \supseteq \cdots$$

forms a chain of compact subsets. Then there is a $z_0 \in \bigcap_{k \in \mathbb{Z}_+} \mathcal{T}^{(k)}$, and since f is holomorphic on Ω, it is holomorphic at z_0. Thus write

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \psi(z)(z - z_0),$$

where $\psi(z) \to 0$ as $z \to z_0$. Integrating both sides over $T^{(n)}$, for any $n \in \mathbb{Z}_+$, yields

$$\int_{T^{(n)}} f(z)\,\mathrm{d}z = f(z_0)\int_{T^{(n)}} \mathrm{d}z + f'(z_0)\int_{T^{(n)}} (z - z_0)\,\mathrm{d}z + \int_{T^{(n)}} \psi(z)(z - z_0),$$

where the first two integrals vanish since the polynomials 1 and $z - z_0$ are entire (and $T^{(n)}$ is closed). So we are left with

$$\int_{T^{(n)}} f(z)\,\mathrm{d}z = \int_{T^{(n)}} \psi(z)(z - z_0).$$

By the ML inequality, we have that

$$\int_{T^{(n)}} f(z)\,\mathrm{d}z \le \sup_{z \in T^{(n)}} |\psi(z)(z - z_0)| \times p^{(n)}.$$

Note that $|z - z_0| \le d^{(n)}$, obviously, and so we have

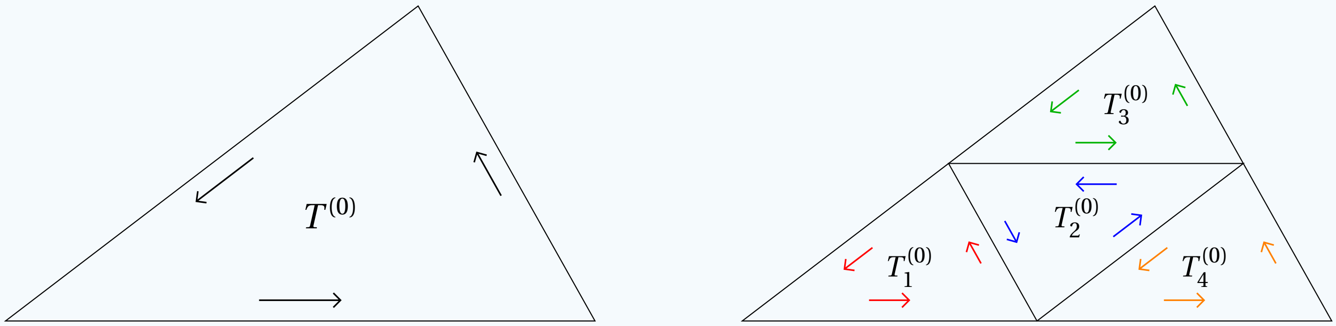
$$\int_{T^{(n)}} f(z)\,\mathrm{d}z \le \varepsilon_n d^{(n)} p^{(n)} = \varepsilon_n d^{(0)} p^{(0)}/4^n,$$

where $\varepsilon_n = \sup_{z \in T^{(n)}} |\psi(z)|$, and since $z \to z_0$ as $n \to \infty$, we have $\varepsilon_n \to 0$ as $n \to \infty$. Finally, we have that

$$\left|\int_{T^{(0)}} f(z)\,\mathrm{d}z\right| \le 4^n\left|\int_{T^{(n)}} f(z)\,\mathrm{d}z\right| \le \varepsilon_n d^{(0)} p^{(0)} \to 0,$$

and so $\displaystyle\int_T f(z)\,\mathrm{d}z = 0$. ❦

[2.0.2] Remark (*Figure for Goursat's Theorem*)



[2.0.3] Corollary

Suppose f is holomorphic on an open set Ω. Let $R \subseteq \Omega$ be some path that traces out a rectangle whose interior also lies in Ω. Then

$$\int_R f(z)\,\mathrm{d}z = 0.$$

Proof. Split the rectangle about the diagonal, yielding two triangles, and apply Goursat's theorem to each. ❦

2.1 Cauchy's Theorem in a Disk

[2.1.1] LEMMA (*Holomorphic Function on Open Disk Admits Primitive*)

Suppose f is holomorphic on the open disk Ω. Then f admits a primitive.

Proof. Without loss of generality, let Ω be centered at the origin. Let $z \in \Omega$ be arbitrary, and consider a path $0 \to \operatorname{Re}(z) \to z$ in the orientation specified, and let it be γ_z.

Then define F by

$$F(z) = \int_{\gamma_z} f(w)\,\mathrm{d}w.$$

We will prove that F is holomorphic on Ω, and that $F'(z) = f(z)$. Fixing $z \in \Omega$, let $h \in \mathbb{C}$ be such that $z + h \in \Omega$ (by virtue of Ω being an open disk). Then write

$$F(z+h) - F(z) = \int_{\gamma_{z+h}} f(w)\,\mathrm{d}w - \int_{\gamma_z} f(w)\,\mathrm{d}w.$$

We may draw out the contours γ_{z+h} and $-\gamma_z$, cancel out the contour from $0 \to \operatorname{Re}(z)$, then draw an auxiliary triangle and yield the resultant contour γ_{res} as a straight line connecting $z, z+h$.

Thus

$$F(z+h) - F(z) = \int_{\gamma_{\text{res}}} f(w)\,\mathrm{d}w.$$

Note that f is continuous at z, so we may write

$$f(w) - f(z) = \psi(w) \implies f(w) = f(z) + \psi(w),$$

where $\psi(z) \to 0$ as $w \to z$. Thus we may write the integral as

$$F(z+h) - F(z) = f(z)\int_{\gamma_{\text{res}}} \mathrm{d}w + \int_{\gamma_{\text{res}}} \psi(w)\,\mathrm{d}w.$$

Obviously, the first integral has the primitive w, and so the first term evaluates to $f(z) \times h$. As for the second integral, we may use the ML inequality to yield

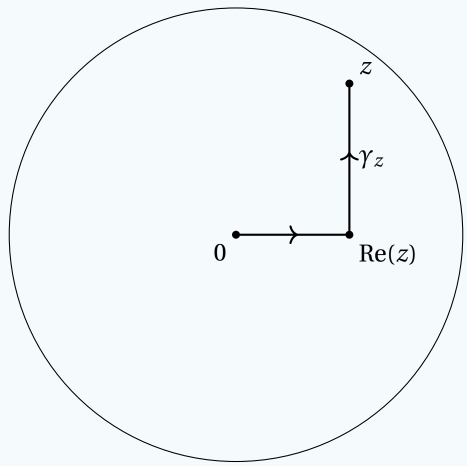
$$\left| \int_{\gamma_{\text{res}}} \psi(w)\,\mathrm{d}w \right| \leq \sup_{w \in \gamma_{\text{res}}} |\psi(w)| \times |h|.$$

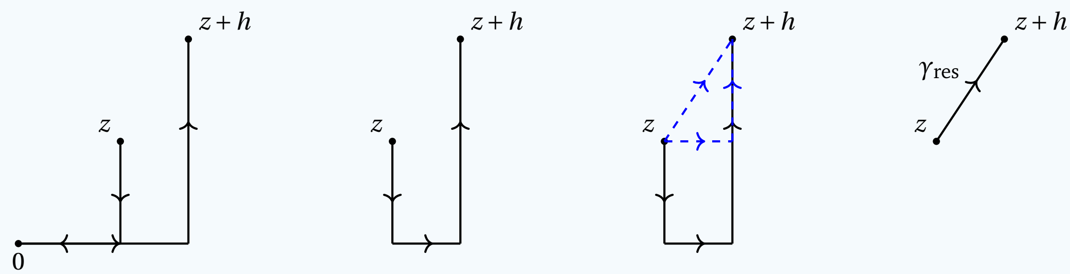
As $h \to 0$, note that supremum goes to 0, and so we have

$$\lim_{h \to 0} \frac{F(z+h) - F(z)}{h} = F'(z) = f(z)$$

by simple rearrangement, completing the proof. ❦

[2.1.2] Remark (*Figures for Cauchy Theorem in a Disk*)





(a) Combined paths (b) After cancellation (c) Auxiliary contours (d) γ_{res}

[2.1.3] Theorem (*Cauchy's Theorem on a Disk*)

Suppose f is holomorphic on some open disk Ω. Then, for any closed curve $\gamma \subseteq \Omega$, we have that

$$\int_\gamma f(z)\,\mathrm{d}z = 0.$$

Proof. f admits a primitive on Ω. ❦

The idea that a holomorphic function on a disk attains a primitive is not unique to disks only. Indeed, the technique of defining our primitive by integrating over a piecewise-smooth path relies purely on our ability to systematically construct polygonal paths, consisting of finitely many horizontal and vertical segments, from a fixed base point z_0 to any point z in the domain without exiting the region of holomorphicity. This geometric framework generalizes naturally to the class of **toy contours**—such as keyholes, sectors, and polygons—where the definition of the "interior" remains intuitive. Within these toy domains, Goursat's theorem for rectangles and triangles guarantees that integrals along any two such polygonal paths agree, yielding a well-defined, single-valued primitive

$$F(z) = \int_{\gamma_z} f(w)\,\mathrm{d}w,$$

which immediately forces the integral over the closed boundary contour to vanish.

However, extending this primitive construction to arbitrary simple closed curves in the plane introduces deep topological difficulties. Defining the interior of a general closed curve rigorously is a non-trivial task, as the curve itself may be highly non-convex or geometrically intricate. Because of the complexity inherent in establishing that the interior of any simple closed curve is well-defined and simply connected, the general proof of Cauchy's theorem for all piecewise-smooth Jordan curves is deferred to Appendix B. There, we will use the topological properties of winding numbers and the Jordan Curve Theorem to establish the equivalence between holomorphic and topological simple connectivity, rigorously completing the generalization of Cauchy's theorem.