1 Preliminaries to Complex Analysis

1.1 Complex Numbers and the Complex Plane

[1.1.1] Definition (*Complex Numbers*)

A **complex number** is an expression of the form $z = x + iy$, where $x, y \in \mathbb{R}$ and $i^2 = -1$.

(1) We write $x = \mathrm{Re}(z)$ for the **real part** of z, and $y = \mathrm{Im}(z)$ for the **imaginary part** of z.

(2) If $\mathrm{Re}(z) = 0$, then z is **purely imaginary**.

(3) The set of all complex numbers is denoted by $\mathbb{C}$.

[1.1.2] Remark (*Vector Space Isomorphism*)

$\mathbb{C}$ is isomorphic to the Euclidean plane $\mathbb{R}^2$ as a real vector space via the canonical bijection:

$$z = x + iy \longleftrightarrow (x, y) \in \mathbb{R}^2$$

Under this identification, the real numbers correspond to the x-axis (the real axis) and the purely imaginary numbers to the y-axis (the imaginary axis).

[1.1.3] Definition (*Absolute Value and Conjugation*)

Let $z = x + iy \in \mathbb{C}$.

(1) The **absolute value** (or modulus) of z is the Euclidean length of its corresponding vector in $\mathbb{R}^2$:

$$|z| = (x^2 + y^2)^{1/2}$$

(2) The **complex conjugate** of z is its reflection across the real axis:

$$\overline{z} = x - iy$$

[1.1.4] Proposition (*Basic Arithmetic and Metric Properties*)

Let $z, z_1, z_2 \in \mathbb{C}$. The following properties hold:

(1) $\mathrm{Re}(z) = \frac{z+\overline{z}}{2}$ and $\mathrm{Im}(z) = \frac{z-\overline{z}}{2i}$.

(2) $|z|^2 = z\overline{z}$.

(3) If $z \neq 0$, then $\frac{1}{z} = \frac{\overline{z}}{|z|^2}$.

(4) **Triangle Inequality**: $|z_1 + z_2| \leq |z_1| + |z_2|$.

(5) **Reverse Triangle Inequality**: $||z_1| - |z_2|| \leq |z_1 - z_2|$.

[1.1.5] Definition (*Polar Form*)

Any non-zero complex number $z \in \mathbb{C}$ can be written in **polar form**:

$$z = re^{i\theta}$$

where $r = |z| > 0$ is the modulus, and $\theta \in \mathbb{R}$ is the **argument** of z (defined uniquely up to a multiple of 2π), denoted by $\arg z$. By Euler's formula:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

[1.1.6] Remark (*Geometric Multiplication*)

If $z = re^{i\theta}$ and $w = se^{i\varphi}$, then $zw = rse^{i(\theta+\varphi)}$. Thus, multiplication by a complex number corresponds geometrically to a homothety in $\mathbb{R}^2$ (a rotation by $\arg z$ composed with a dilation by $|z|$).

1.2 Convergence

[1.2.1] Definition (*Convergence of Sequences*)

A sequence $\{z_n\} \subset \mathbb{C}$ **converges** to $w \in \mathbb{C}$ if:

$$\lim_{n\to\infty} |z_n - w| = 0$$

We write $w = \lim_{n\to\infty} z_n$.

[1.2.2] Proposition (*Component-wise Convergence*)

A sequence $\{z_n\} \subset \mathbb{C}$ converges to w if and only if the sequences of real and imaginary parts of z_n converge to the real and imaginary parts of w, respectively.

[1.2.3] Definition (*Cauchy Sequence*)

A sequence $\{z_n\} \subset \mathbb{C}$ is a **Cauchy sequence** if for every $\varepsilon > 0$, there exists an integer $N > 0$ such that:

$$|z_n - z_m| < \varepsilon \quad \text{whenever } n, m > N$$

[1.2.4] Theorem (*Completeness of* $\mathbb{C}$)

The complex numbers $\mathbb{C}$ form a complete metric space. That is, every Cauchy sequence in $\mathbb{C}$ converges to a limit in $\mathbb{C}$.

> **Proof.** Since $\{z_n\}$ is Cauchy if and only if the component sequences $\{\mathrm{Re}(z_n)\}$ and $\{\mathrm{Im}(z_n)\}$ are Cauchy in $\mathbb{R}$, completeness of $\mathbb{C}$ follows directly from the completeness of $\mathbb{R}$. ❦

1.3 Sets in the Complex Plane

[1.3.1] Definition (*Discs*)

Let $z_0 \in \mathbb{C}$ and $r > 0$.

(1) The **open disc** of radius r centered at z_0 is:

$$D_r(z_0) = \{z \in \mathbb{C} : |z - z_0| < r\}$$

(2) The **closed disc** of radius r centered at z_0 is:

$$\overline{D}_r(z_0) = \{z \in \mathbb{C} : |z - z_0| \le r\}$$

(3) The boundary of either diskis the circle $C_r(z_0) = \{z \in \mathbb{C} : |z - z_0| = r\}$.

(4) The **unit disc** is denoted by $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

[1.3.2] Definition (*Topological Vocabulary*)

Let $\Omega \subset \mathbb{C}$ be a set.

(1) A point $z_0 \in \Omega$ is an **interior point** of Ω if there exists $r > 0$ such that $D_r(z_0) \subset \Omega$.

(2) The **interior** of Ω, denoted Ω°, is the set of all its interior points.

(3) Ω is **open** if every point in Ω is an interior point (i.e., $\Omega = \Omega^\circ$).

(4) Ω is **closed** if its complement $\Omega^c = \mathbb{C} \setminus \Omega$ is open.

(5) $z \in \mathbb{C}$ is a **limit point** of Ω if there is a sequence $\{z_n\} \subset \Omega$ with $z_n \neq z$ such that $z_n \to z$. A set is closed iff it contains all its limit points.

(6) The **closure** of Ω is $\overline{\Omega} = \Omega \cup \{\text{limit points}\}$.

(7) The **boundary** of Ω is $\partial\Omega = \overline{\Omega} \setminus \Omega^\circ$.

(8) Ω is **bounded** if there exists $M > 0$ such that $|z| < M$ for all $z \in \Omega$.

(9) If Ω is bounded, its **diameter** is:

$$\operatorname{diam}(\Omega) = \sup_{z,w\in\Omega} |z - w|$$

[1.3.3] Definition (*Compactness*)

A set $\Omega \subset \mathbb{C}$ is **compact** if it is closed and bounded.

[1.3.4] Theorem (*Characterizations of Compactness*)

Let $\Omega \subset \mathbb{C}$ be a subset. The following are equivalent:

(1) Ω is compact (closed and bounded).

(2) Every sequence $\{z_n\} \subset \Omega$ has a subsequence that converges to a point in Ω (sequential compactness).

(3) Every open covering of Ω admits a finite subcovering.

[1.3.5] Proposition (*Nested Compact Sets*)

If $\Omega_1 \supset \Omega_2 \supset \cdots \supset \Omega_n \supset \cdots$ is a nested sequence of non-empty compact sets in $\mathbb{C}$ with $\operatorname{diam}(\Omega_n) \to 0$ as $n \to \infty$, then there exists a unique point $w \in \mathbb{C}$ such that $w \in \Omega_n$ for all n.

[1.3.6] Definition (*Connectedness and Regions*)

An open set $\Omega \subset \mathbb{C}$ is **connected** if it is not possible to find two disjoint, non-empty open sets Ω_1, Ω_2 such that $\Omega = \Omega_1 \cup \Omega_2$.

(1) A connected open set in $\mathbb{C}$ is called a **region**.

(2) For open sets in $\mathbb{C}$, connectedness is equivalent to **path-connectedness**: any two points in Ω can be joined by a continuous curve entirely contained in Ω.

1.4 Continuous Functions

[1.4.1] Definition (*Continuity*)

Let f be a complex-valued function defined on $\Omega \subset \mathbb{C}$.

(1) f is **continuous** at $z_0 \in \Omega$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that:

$$|f(z) - f(z_0)| < \varepsilon \quad \text{whenever } z \in \Omega \text{ and } |z - z_0| < \delta$$

(2) Equivalently, f is continuous at z_0 if for every sequence $z_n \to z_0$ in Ω, we have $f(z_n) \to f(z_0)$.

[1.4.2] Theorem (*Extreme Value Theorem*)

A continuous function f on a compact set $\Omega \subset \mathbb{C}$ is bounded and attains both a maximum and a minimum value on Ω (where maximum/minimum are defined in terms of the real-valued modulus $|f(z)|$).

Note that there is no analogue of the intermediate value theorem, since $\mathbb{C}$ does not admit a standard ordering that respects its structure.

1.5 Holomorphic Functions

[1.5.1] Definition (*Holomorphicity*)

Suppose $\Omega \subseteq \mathbb{C}$ is an open set with $z_0 \in \Omega$. Then a function f is said to be **holomorphic at** z_0 if the value

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} \qquad h \in \mathbb{C}$$

exists.

A function f is said to be **holomorphic** on Ω if it is holomorphic at every point in Ω. For a general, non-open set $C \subseteq \mathbb{C}$, we say f is holomorphic on C if it is holomorphic on some open set containing C.

Note that every point in an open set is an interior point, and so the open disk$D_r(z_0) \subseteq \Omega$ for some small $r > 0$ (ensuring $z_0 + h \in \Omega$ for all complex h with $|h| < r$, which prevents vacuous edge-cases). This is the first definition that is truly *complex* in nature. While this resembles the real definition of differentiability, note that h is not real-valued, but rather $h \in (\mathbb{C} \cong \mathbb{R}^2)$. We have made complex differentiability far more strict: this limit must exist not just when approaching from the left and right as in $\mathbb{R}$, but along any path, from every available angle, as $h \to 0$ in the complex plane. Of course, also note that the limit that defines $f'(z_0)$ is referred to as the (complex) derivative of f at z_0.

[1.5.2] Theorem (*Holomorphicity Implies Continuity*)

Suppose f is holomorphic at some z_0 in f's domain. Then f is continuous at z_0.

Proof. We show $f(z_0 + h) \to f(z_0)$ as $h \to 0$ as such:

$$\lim_{h \to 0} f(z_0 + h) - f(z_0) = \lim_{h \to 0} \left(h \times \frac{f(z_0 + h) - f(z_0)}{h} \right) = 0 \times f'(z_0) = 0.$$

❦

Rearranging our definition of a complex derivative, we have that

$$\lim_{h\to 0}\frac{f(z_0+h)-f(z_0)}{h}=a \implies \underbrace{[f(z_0+h)-f(z_0)]}_{\Delta f}-ah=o(h),$$

where $o(h)$ is some map such that $\lim_{h\to 0} o(h)/h=0$. This statement says that, for $h\in\mathbb{C}$ of small magnitude, we can approximate the deviation of f from z_0 (denoted $\Delta f = f(z_0+h)-f(z_0)$) very well with the linear change ah. This approximation is so good that the error goes to zero much quicker than h can go to 0. Note that limits (and thus derivatives) are unique, and so a is the unique constant that approximates Δf by the linear function ah. In multivariate theory, we say that the linear map $h\mapsto ah$ is the true derivative of f, and this idea can be extended to all forms of finite-dimensional domains and codomains.

[1.5.3] PROPOSITION (*Basic Differentiation Rules for* $\mathbb{C}$)

Suppose f, g are holomorphic functions on Ω.

(1) $f+g$ is holomorphic in Ω, with $(f+g)'=f'+g'$.

(2) fg is holomorphic in Ω, with $(fg)'=f'g+fg'$.

(3) f/g is holomorphic in Ω at points for which $g\neq 0$, with $(f/g)'=(f'g-fg')/g^2$.

(4) The chain rule holds. For the two-function case $f:\Omega\to U$ and $g:U\to\mathbb{C}$, where both are holomorphic, we have that

$$(g\circ f)'=g'(f(z))\,f'(z)\quad \forall z\in\Omega.$$

From this, we can prove basic results that polynomials, rational functions, and other nice classes of functions are holomorphic for suitable domains.

1.6 Complex-Valued Functions as Mappings

[1.6.1] REMARK (*Real Differentiability vs. Complex Differentiability*)

We now make the distinction between treating a complex-valued function as a function $\mathbb{R}^2\to\mathbb{R}^2$. Consider the function $f(z)=\overline{z}$, then note

$$f'(z)=\lim_{h\to 0}\frac{f(z+h)-f(z)}{h}=\lim_{h\to 0}\frac{\overline{h}}{h},$$

which does not exist (parameterizing $\overline{z}/z$ by an angle yields $\exp(-2i\theta)$, and so the limit varies based on path chosen). Thus $f:\mathbb{C}\to\mathbb{C}$ is not holomorphic. But now consider $g:\mathbb{R}^2\to\mathbb{R}^2$ defined by $g(x,y)=(u(x,y),v(x,y))=(x,-y)$. This function, as a real-valued one, is linear, infinitely differentiable, and enjoys many properties. One such property is that its total derivative (as a linear map), expressed in the standard basis of $\mathbb{R}^2$, is written as

$$g'=\begin{bmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}.$$

From the perspective of multivariate theory, a map $g:\mathbb{R}^2\to\mathbb{R}^2$ has a derivative that is a linear map in two-dimensions, in contrast with a map $f:\mathbb{C}\to\mathbb{C}$ having a singular, complex number for its derivative (that is also a one-dimensional linear map).

[1.6.2] Remark (*Cauchy-Riemann Equations*)

We can link real differentiability and complex differentiability, however. For a complex function $f : \Omega \to \mathbb{C}$ to be differentiable, its difference function must take on the same value as $h \to 0$ through any path. Writing $h = h_1 + ih_2$ and $z = x + iy$ (and so $f(z) = f(x, y)$), we may choose a point $z_0 = x_0 + iy_0 \in \Omega$ for which $f'(z_0)$ exists and yield the following:

$$\begin{aligned} f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} &= \lim_{h_1 \to 0} \frac{f(x_0 + h_1, y_0) - f(x_0, y_0)}{h_1} = \frac{\partial f}{\partial x}(x_0) \quad (h_2 = 0) \\ &= \lim_{h_2 \to 0} \frac{f(x_0, y_0 + h_2) - f(x_0, y_0)}{ih_2} = \frac{1}{i}\frac{\partial f}{\partial y}(x_0) \quad (h_1 = 0). \end{aligned}$$

If f is holomorphic, we have that $\frac{\partial f}{\partial x} = \frac{1}{i}\frac{\partial f}{\partial y} = -i\frac{\partial f}{\partial y}$ by the uniqueness of the derivative. If we write $f(z) = u(z) + iv(z)$, then we have that the partials u_x, u_y, v_x, v_y all exist, and they are such that

$$\frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y} \implies \left(\frac{\partial u}{\partial x}\right) + i\left(\frac{\partial v}{\partial x}\right) = i\left[\left(\frac{\partial u}{\partial y}\right) + i\left(\frac{\partial v}{\partial y}\right)\right] = \left(-\frac{\partial v}{\partial y}\right) + i\left(\frac{\partial u}{\partial y}\right),$$

implying the **Cauchy-Riemann equations**:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

[1.6.3] Remark (*Derivation of Wirtinger Operators*)

We will motivate the introduction of two important differential operators known as Wirtinger operators. To get started, let $z \in \mathbb{C}$ be a complex variable. Noting that $\mathbb{C} \cong \mathbb{R}^2$, write $z = x + iy$ and $\overline{z} = x - iy$. We may "solve" for x, y in terms of $z, \overline{z}$ as such:

$$x = \frac{z + \overline{z}}{2} \qquad y = \frac{z - \overline{z}}{2i}.$$

Treating $\overline{z}$ as independent of z, we can expand the differential operators $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \overline{z}}$ by the multivariate chain rule.

$$\frac{\partial}{\partial z} = \underbrace{\frac{\partial x}{\partial z}}_{1/2}\frac{\partial}{\partial x} + \underbrace{\frac{\partial y}{\partial z}}_{1/2i}\frac{\partial}{\partial y} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}\right),$$

$$\frac{\partial}{\partial \overline{z}} = \underbrace{\frac{\partial x}{\partial \overline{z}}}_{1/2}\frac{\partial}{\partial x} + \underbrace{\frac{\partial y}{\partial \overline{z}}}_{-1/2i}\frac{\partial}{\partial y} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \frac{1}{1}\frac{\partial}{\partial y}\right).$$

[1.6.4] Proposition

If $f(z) = u(z) + iv(z)$ is holomorphic at z_0, then

$$\frac{\partial f}{\partial \bar{z}}(z_0) = 0 \quad f'(z_0) = \frac{\partial f}{\partial z}(z_0) = 2\frac{\partial u}{\partial z}(z_0).$$

Let F be a mapping in terms of real variables such that $F(x, y) = f(x+iy)$. Then we have that F is differentiable, and its derivative can be represented by a Jacobian J_F such that

$$\det\big(J_F(x_0, y_0)\big) = |f(x_0 + iy_0)|^2.$$

Proof. Suppose f is holomorphic at z_0. Then f satisfies the Cauchy-Riemann equations, meaning $\frac{\partial f}{\partial x}(z_0) = \frac{1}{i}\frac{\partial f}{\partial y}$, and so

$$\frac{\partial f}{\partial \bar{z}}(z_0) = \frac{1}{2}\left(\frac{\partial f}{\partial x}(z_0) - \frac{1}{i}\frac{\partial f}{\partial y}(z_0)\right) = 0.$$

From this, we have that

$$f'(z_0) = \frac{1}{2}\left(\frac{\partial f}{\partial x}(z_0) + \frac{\partial f}{\partial x}(z_0)\right) = \frac{1}{2}\left(\frac{\partial f}{\partial x}(z_0) + \frac{1}{i}\frac{\partial f}{\partial y}(z_0)\right) = \frac{\partial f}{\partial z}(z_0).$$

From the Cauchy-Riemann Equations, we get that $\frac{\partial f}{\partial z} = 2\frac{\partial u}{\partial z}$ for $f = u + iv$. We have already proved that complex differentiability implies real differentiability by the Cauchy-Riemann equations. Finally, note that

$$J_F(x_0, y_0) = \begin{bmatrix} u_x(x_0) & u_y(y_0) \\ v_x(x_0) & v_y(y_0) \end{bmatrix},$$

and so $\det\big(J_F(x_0, y_0)\big) = u_x(x_0)v_y(y_0) - u_y(y_0)v_x(x_0) = (u_x(x_0))^2 + (u_y(y_0))^2 = |f'(z_0)|^2$.

[1.6.5] Theorem (*Criterion for Holomorphicity*)

A function $f(z) = u(z) + iv(z)$ is **holomorphic** if and only if u, v have continuous, first-order partial derivatives that satisfy the Cauchy-Riemann equations.

Proof.

$(\Longrightarrow)$: This is implied from taking the limit of the difference quotient along $\mathbb{R}$ and $i\mathbb{R}$.

$(\Longleftarrow)$: Suppose u, v have continuous first-order partial derivatives that satisfy the Cauchy-Riemann equations. From multivariate theory, note that

$$u(x+h, y+k) - u(x, y) = u_x h + u_y k + o_1(h+ik),$$
$$v(x+h, y+k) - v(x, y) = v_x h + v_y k + o_2(h+ik).$$

Then note

$$f(z + [h+ik]) - f(z) = (u_x + iv_x)(h+ik) + [o_1(h+ik) + io_2(h+ik)],$$

meaning

$$f'(z) = \lim_{h+ik \to 0} \frac{f(z+[h+ik]) - f(z)}{h+ik} = u_x + iv_x.$$

1.7 Power Series

[1.7.1] Definition (*Complex Exponential and Trigonometric Functions*)

The complex **exponential function** is defined for all $z \in \mathbb{C}$ by the power series:

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

This series converges absolutely for every $z \in \mathbb{C}$, and converges uniformly in every closed diskin $\mathbb{C}$.

Using the complex exponential, the standard complex **trigonometric functions** are defined by:

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} = \frac{e^{iz} + e^{-iz}}{2},$$

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} = \frac{e^{iz} - e^{-iz}}{2i}.$$

These definitions yield the **Euler formulas** and agree with their real counterparts for $z \in \mathbb{R}$.

[1.7.2] Theorem (*Hadamard's Formula and Convergence*)

Given a power series $\sum_{n=0}^{\infty} a_n z^n$, there exists a **radius of convergence** R $(0 \le R \le \infty)$ such that:

(1) If $|z| < R$, the series converges absolutely.

(2) If $|z| > R$, the series diverges.

Under the conventions $1/0 = \infty$ and $1/\infty = 0$, R is determined by **Hadamard's formula**:

$$1/R = \limsup_{n \to \infty} |a_n|^{1/n}.$$

The open region $D_R(0) = \{z \in \mathbb{C} : |z| < R\}$ is called the **disk of convergence**.

Proof. Let $L = 1/R = \limsup_{n \to \infty} |a_n|^{1/n}$.

If $|z| < R$, then $L|z| < 1$. Choose $\varepsilon > 0$ sufficiently small such that $(L + \varepsilon)|z| = r < 1$. By the definition of the limit superior, we have $|a_n|^{1/n} \le L + \varepsilon$ for all sufficiently large n. Thus:

$$|a_n||z|^n \le [(L + \varepsilon)|z|]^n = r^n.$$

Since $r < 1$, absolute convergence follows by comparison with the convergent geometric series $\sum r^n$.

If $|z| > R$, then $L|z| > 1$. A similar argument shows that $|a_n||z|^n > 1$ infinitely often, so the terms do not tend to 0, forcing divergence. ❦

[1.7.3] Remark

The behavior of a power series on its boundary $|z| = R$ is delicate; one can have either convergence or divergence depending on the specific coefficients (e.g., $\sum z^n$, $\sum z^n/n^2$, and $\sum z^n/n$ on the unit circle $\partial\mathbb{D}$).

[1.7.4] Theorem (*Holomorphicity of Power Series*)

The power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ defines a holomorphic function in its disk of convergence. Furthermore, $f'(z)$ can be obtained by term-by-term differentiation:

$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1},$$

and this differentiated series has the same radius of convergence R.

[1.7.5] Corollary (*Infinite Complex Differentiability*)

A power series $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ is infinitely complex differentiable in its disk of convergence, and all higher-order derivatives are obtained by term-by-term differentiation.

[1.7.6] Definition (*Analytic Functions*)

A complex-valued function f defined on an open set $\Omega \subseteq \mathbb{C}$ is said to be **analytic** (or to have a **power series expansion**) at $point$ $z_0 \in \Omega$ if there exists a power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ with a positive radius of convergence such that:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad \text{for all } z \text{ in a neighborhood of } z_0.$$

If f is analytic at every point in Ω, we say f is **analytic on** Ω.

[1.7.7] Remark

By Theorem 2.6, any analytic function on Ω is automatically holomorphic on Ω. A fundamental and deep result in complex analysis (the converse, which we prove in Chapter 2) is that every holomorphic function is analytic. Thus, we will use the terms *holomorphic* and *analytic* interchangeably.

1.8 Integration along Curves

[1.8.1] Definition (*Parameterized Curves, Smooth Curves, Piecewise-Smooth Curves*)

A **parameterized curve** is some function $z(t)$ that maps $[a,b] \subseteq \mathbb{R}$ into the complex plane $\mathbb{C}$. We say that the parameterized curve is **smooth** if $z'(t)$ exists, is continuous, and is never equal to 0 for every $t \in [a,b]$. We say that the parameterized curve is **piecewise-smooth** if z is continuous on $[a,b]$ and has a partition

$$a = a_0 < a_1 < \cdots < a_{n-1} < a_n = b$$

such that $z(t)$ is smooth when restricted to any subinterval $[a_k, a_{k+1}]$. Note that this definition does not require the left and right-hand derivatives for z' to agree at each a_k for $k \in \{1, \ldots, n-1\}$.

For any circle of radius r centered at z_0 in the complex plane, we note that $z(t) = z_0 + re^{it}$ for $t \in [0, 2\pi]$ is a smooth parameterization of it. But note that $z_0 + re^{2it}$ parameterizes the same curve for the same domain, and so does $z_0 + re^{4it}$, and infinitely more parameterizations. We seek a notion of equality between these curves.

[1.8.2] Definition (*Equivalence of Parameterized Curves*)

Two parameterizations $z : [a,b] \to \mathbb{C}$ and $\tilde{z} : [c,d] \to \mathbb{C}$ are **equivalent** if there exists a continuously differentiable map $[c,d] \to [a,b]$ defined by $s \mapsto t(s)$ such that $\tilde{z}(s) = z(t(s))$ and $t'(s) > 0$ for every $s \in [c,d]$.

Note the condition that $t'(s) > 0$ in the previous definition ensures that as s ticks forward, $t(s)$ also ticks forward, meaning z can never change direction. Thus equivalent curves share the same direction/orientation, and so a curve and its reverse aren't equivalent curves.

Equality in this sense is an equivalence relation, and so equivalent curves form equivalence classes. When referring to a curve γ with no explicit parameterization, we refer to the entire equivalence class of functions that can parameterize γ. That is, a curve represents the set of all valid paths tracing the exact same geometric set of points with the same orientation (that is, γ and $-\gamma$ are not necessarily equal).

[1.8.3] Definition (*Closed and Simple Curves*)

A smooth curve parameterized by $z : [a,b] \to \mathbb{C}$ is called **closed** if $z(a) = z(b)$. A curve is called **simple** if $z(s) \neq z(t)$ for any $s \neq t$ (i.e., the curve never self-intersects). A closed, simple curve is a closed curve that is simple when disregarding its endpoints.

[1.8.4] Definition (*Integral Along Smooth and Piecewise-Smooth Curve*)

Suppose γ is a smooth curve, and let f be a continuous function on γ. Let $z : [a,b] \to \mathbb{C}$ be any parameterization of γ, and define the **integral of f along** γ by

$$\int_\gamma f(z)\,\mathrm{d}z = \int_a^b f(z(t))\,z'(t)\,\mathrm{d}t.$$

If γ is piecewise-smooth, let $z : [a,b] \to \mathbb{C}$ be any piecewise-smooth parameterization over the partition $a = a_0 < a_1 < \cdots < a_n = b$. Then the integral of f along γ is defined by

$$\int_\gamma f(z)\,\mathrm{d}z = \sum_{k=0}^{n-1} \int_{a_k}^{a_{k+1}} f(z(t))\,z'(t)\,\mathrm{d}t.$$

[1.8.5] Theorem (*Integral Along Curve Does Not Depend on Parameterization*)

Suppose γ is a smooth curve. Then the integral of f along γ is independent of parameterization.

Proof. Let $z : [a,b] \to \mathbb{C}$ and $\tilde{z} : [c,d] \to \mathbb{C}$ be equivalent parameterizations of γ. Thus there is some continuously bijective map $s \mapsto t(s)$ such that $t'(s) > 0$ and $\tilde{z}(s) = z(t(s))$ for every $s \in [c,d]$. Then note

$$\begin{aligned}
\int_\gamma f(z)\,\mathrm{d}z &= \int_a^b f(z(t))\,z'(t)\,\mathrm{d}t \\
&= \int_c^d f(z(t(s)))\,z'(t(s))\;t'(s)\,\mathrm{d}s && \text{[Change of Variables]} \\
&= \int_c^d f(\tilde{z}(s))\,\tilde{z}(s)\,\mathrm{d}s && \text{[Multivariate Chain Rule]} \\
&= \int_\gamma f(z)\,\mathrm{d}z.
\end{aligned}$$

❦

[1.8.6] Definition (*Length of Curve*)

Let γ be a smooth curve with a parameterization $z : [a,b] \to \mathbb{C}$. Then the **length** of γ is given by

$$\operatorname{length}(\gamma) = \int_\gamma \mathrm{d}s = \int_a^b |z'(t)|\,\mathrm{d}t.$$

[1.8.7] Proposition (*Properties of Contour Integrals*)

Suppose f, g are continuous functions and γ is a smooth curve.

(1) **Linearity.** $\displaystyle\int_\gamma (af(z)+bg(z))\,\mathrm{d}z = a\int_\gamma f(z)\,\mathrm{d}z + b\int_\gamma g(z)\,\mathrm{d}z.$

(2) **Path Reversal.** $\displaystyle\int_\gamma f(z)\,\mathrm{d}z = -\int_{-\gamma} f(z)\,\mathrm{d}z.$

(3) **ML/Estimation Lemma.** $\displaystyle\left|\int_\gamma f(z)\,\mathrm{d}z\right| \le \sup_{z\in\gamma}|f(z)| \times \mathrm{length}(\gamma).$

Proof.

(1) Follows from definition.

(2) Let $z_1 : [a,b] \to \mathbb{C}$ be a parameterization for γ, and let $z_2 : [b,a] \to \mathbb{C}$ be a parameterization for $-\gamma$ such that $z_2(t) = z_1(b+a-t)$. Then note $z_2'(t) = -z_1'(b+a-t)$. Thus we yield

$$\int_\gamma f(z)\,\mathrm{d}z = \int_a^b f(z_1(t))\,z_1'(t)\,\mathrm{d}t = \int_b^a f(z_1(b+a-t))\,z_1'(b+a-t)\,\mathrm{d}t = -\int_b^a f(z_2(t))\,z_2'(t)\,\mathrm{d}t = -\int_{-\gamma} f(z)\,\mathrm{d}z.$$

(3) Let $z : [a,b] \to \mathbb{C}$ be a parameterization of γ.

$$\left|\int_\gamma f(z)\,\mathrm{d}z\right| = \left|\int_a^b f(z(t))\,z'(t)\,\mathrm{d}t\right| = \int_a^b \big|f(z(t))\big|\big|z'(t)\big|\,\mathrm{d}t \le \sup_{t\in[a,b]} |f(z(t))| \int_a^b |z'(t)|\,\mathrm{d}t = \sup_{z\in\gamma}|f'(z)| \times \mathrm{length}(\gamma).$$

❦

[1.8.8] Definition (*Primitive*)

Suppose f is a function on Ω. Then a function F that is holomorphic on Ω is called a **primitive** (or an **antiderivative**) of f if $F'(z) = f(z)$ for every $z \in \Omega$.

[1.8.9] Theorem (*FToC Applied to Contour Integration*)

If f is a continuous function on Ω that has a primitive F, and $\gamma \subseteq \Omega$ is a path from w_1 to w_2, we have that

$$\int_\gamma f(z)\,\mathrm{d}z = F(w_2) - F(w_1).$$

Proof. Suppose γ is smooth. Then let $z : [a,b] \to \mathbb{C}$ be a smooth parameterization of γ, and note

$$\begin{aligned}\int_\gamma f(z)\,\mathrm{d}z &= \int_a^b f(z(t))\,z'(t)\,\mathrm{d}t \\ &= \int_a^b F'(z(t))\,z'(t)\,\mathrm{d}t \\ &= \int_a^b \frac{\mathrm{d}}{\mathrm{d}x}F(z(t))\mathrm{d}t \\ &= F(z(b)) - F(z(a)) = F(w_2) - F(w_1).\end{aligned}$$

For γ piecewise-smooth, the exact same process occurs, but instead of an integral we use a sum over each partition. ❦

[1.8.10] Corollary (*Integral About Closed Contour*)

If γ is a closed curve in Ω, and f is continuous and has a primitive in Ω, then

$$\int_\gamma f(z)\,\mathrm{d}z = 0.$$

Note that $f(z) = 1/z$ does not admit a primitive in the punctured complex plane $\mathbb{C}^* = \mathbb{C}\setminus\{0\}$. Indeed, if we use the unit circle parameterized by $z = e^{it}$ for $t \in [0, 2\pi]$, we have that

$$\int_C \frac{1}{z}\,\mathrm{d}z = \int_0^{2\pi} \frac{1}{e^{it}} \times ie^{it}\,\mathrm{d}t = 2\pi i \neq 0,$$

and so $1/z$ does not have a primitive by contrapositive argumentation.

[1.8.11] Corollary

If f is holomorphic in a region Ω and $f' = 0$, then f is constant.

> **Proof.** Fix a point $w_0 \in \Omega$. Since Ω is a connected subset of $\mathbb{C}$, it is path-connected, so let $\gamma \subseteq \Omega$ be any path that terminates at w_0. Note then that
>
> $$0 = \int_\gamma f'(z) = f(w_0) - f(w),$$
>
> and so $f(w_0) = f(w)$. Since w is arbitrary, we have that f is constant on Ω. ❦

1.8.1 Exercises

[1.8.12] Problem

(1) Let γ be any circle about the origin. Evaluate $\displaystyle\int_\gamma z^n$ for all integers n.

(2) Evaluate the integral for any circle that is not centered at 0.

(3) Show that if $|a| < r < |b|$, then

$$\int_\gamma \frac{1}{(z-a)(z-b)}\,\mathrm{d}z = \frac{2\pi i}{b-a},$$

where γ is the circle of radius r about the origin.

Proof.

(1) Let γ be any circle about the origin. For simplicity, suppose $r = 1$, and let $z(t) = e^{it}$ be a parameterization of γ over $[0, 2\pi]$. Then note

$$\int_\gamma z^n\,\mathrm{d}z = \int_0^{2\pi} (e^{int})(ie^{it})\,\mathrm{d}t = i\int_0^{2\pi} e^{it(n+1)}\,\mathrm{d}t = \frac{1}{n+1}\left[e^{it(n+1)}\right]_{t=0}^{t=2\pi} = \frac{e^{2\pi i(n+1)}-1}{n+1} = \frac{1^{n+1}-1}{n+1} = 0$$

for all $n \neq -1$. Note that if we let r be arbitrary, then the result would simply be the integral scaled by r^n, which is still 0. In the case of $n = -1$, we have that

$$\int_\gamma \frac{1}{z}\,\mathrm{d}z = \int_0^{2\pi} \frac{i(re^{it})}{re^{it}} = 2\pi i.$$

(2) Note that z^n is holomorphic on $\mathbb{C}^*$ for all $n \in \mathbb{Z}_+$. Let Ω be an open set such that $\gamma \subseteq \Omega$, and so $\displaystyle\int_\gamma z^n\,\mathrm{d}z = 0$ by holomorphicity (and thus continuity) of the integrand and by γ being a closed curve.

(3) Split the integrand by the method of partial fractions:

$$\int_\gamma \frac{1}{(z-a)(z-b)}\,\mathrm{d}z = \underbrace{\left(\frac{1}{a-b}\int_\gamma \frac{1}{z-a}\,\mathrm{d}z\right)}_{:=I_1} + \underbrace{\left(\frac{1}{b-a}\int_\gamma \frac{1}{z-b}\,\mathrm{d}z\right)}_{:=I_2}.$$

I_2 is easy to evaluate.

❦

[1.8.13] Problem

Suppose f is continuous on a region Ω. Prove that if f has a primitive on Ω, then it is unique.

Proof. Suppose f admits two primitives F_1, F_2 on Ω. Fix $w_0 \in \Omega$, and let $w \in \Omega$ vary. Since Ω is a region, it is path-connected, so suppose γ is a path from w to w_0. Then note

$$\int_\gamma f(z)\,\mathrm{d}z = F_1(w_0) - F_1(w) = F_2(w_0) - F_2(w) \implies F_1(w) - F_2(w) = \underbrace{F_1(w_0) - F_2(w_0)}_{:=C}.$$

Thus $F_1 - F_2 \equiv C$, and so these functions are equivalent up to a constant.

❦

2 Cauchy's Theorem and Its Applicatins

We have proven that if f is continuous and admits a primitive F on an open set Ω, then for any path $\gamma \subseteq \mathbb{C}$ that starts at w and ends at w_0, we have that

$$\int_\gamma f(z)\,\mathrm{d}z = F(w_0) - F(w).$$

From this, we immediately have that if γ is a closed path, then the integral evaluates to 0.

This mirrors the fundamental theorem of calculus, with the exception that this doesn't apply to all continuous functions—the condition that f admits a primitive is crucial. In real variables, a continuous function automatically admits a primitive/antiderivative, but this is not necessarily true in complex variables. For example, consider the function $1/z$ that is entire on $\mathbb{C}^*$. For the unit circle γ, note that

$$\int_\gamma \frac{1}{z}\,\mathrm{d}z = \int_0^{2\pi} \frac{ie^{it}}{e^{it}}\,\mathrm{d}t = 2\pi i \neq 0,$$

and so by contrapositive, $1/z$ has no primitive on $\mathbb{C}^*$ in spite of its holomorphicity.

Explicitly finding a primitive for a function is admittedly difficult. We alter our hypotheses to be more tractable, yielding **Cauchy's theorem**. It states that if f is holomorphic on an open set Ω, then for any path $\gamma \subseteq \Omega$ whose interior is also contained on Ω, we have that

$$\int_\gamma f(z)\,\mathrm{d}z = 0.$$

We prove this for toy contours (i.e., for contours where the "interior" is intuitively obvious) and defer the proof of the full statement for simply closed (i.e., Jordan) curves to the future. We start by proving Goursat's theorem, which proves this statement for the case where γ is a triangle.

[2.0.1] Theorem (*Goursat's Theorem*)

Suppose f is holomorphic on an open set Ω. Let $T \subseteq \Omega$ be some path that traces out a triangle whose interior also lies in Ω. Then

$$\int_T f(z)\,\mathrm{d}z = 0.$$

Proof. Let $T^{(0)}$ be the triangle in question, whose orientation is fixed and considered "positive," and let $d^{(0)}, p^{(0)}$ represent the diameter and perimeter of these triangles respectively. Consider the midpoints of each side of $T^{(0)}$ and connect each of them by a straight line, partitioning $T^{(0)}$ into four triangles $T_k^{(0)}$ for $k = 1,2,3,4$. We choose the direction of each triangle in the diagram shown in the next remark block.

Since the contours that aren't on the sides of $T^{(0)}$ cancel out, we know that

$$\int_{T^{(0)}} f(z)\,\mathrm{d}z = \sum_{k=1}^{4} \int_{T_k^{(0)}} f(z)\,\mathrm{d}z.$$

By the triangle inequality, there exists a $k \in \{1,2,3,4\}$ such that

$$\left|\int_{T^{(0)}} f(z)\,\mathrm{d}z\right| \le 4\left|\int_{T_k^{(0)}} f(z)\,\mathrm{d}z\right|.$$

Let $T^{(1)} = T_k^{(0)}$, and then let $d^{(1)}, p^{(1)}$ be the diameter and perimeter for $T^{(1)}$. By geometry, we note that $d^{(1)} = d^{(0)}/2$ and $p^{(1)} = p^{(0)}/2$. We generate a sequence of triangles

$$T^{(0)}, T^{(1)}, \ldots, T^{(n)}, \ldots$$

with the properties that

$$\left|\int_{T^{(0)}} f(z)\,\mathrm{d}z\right| \le 4^n\left|\int_{T^{(n)}} f(z)\,\mathrm{d}z\right|, \quad d^{(n)} = d^{(0)}/2^n, \quad p^{(n)} = p^{(0)}/2^n.$$

Let $\mathcal{T}^{(k)} = \overline{T^{(k)}}$, a closed and bounded (and thus compact) subspace of $\mathbb{C}$ in its metric topology. Thus

$$\mathcal{T}^{(0)} \supseteq \mathcal{T}^{(1)} \supseteq \cdots$$

forms a chain of compact subsets. Then there is a $z_0 \in \bigcap_{k\in\mathbb{Z}_+} \mathcal{T}^{(k)}$, and since f is holomorphic on Ω, it is holomorphic at z_0. Thus write

$$f(z) = f(z_0) + f'(z_0)(z-z_0) + \psi(z)(z-z_0),$$

where $\psi(z) \to 0$ as $z \to z_0$. Integrating both sides over $T^{(n)}$, for any $n \in \mathbb{Z}_+$, yields

$$\int_{T^{(n)}} f(z)\,\mathrm{d}z = f(z_0)\int_{T^{(n)}} \mathrm{d}z + f'(z_0)\int_{T^{(n)}} (z-z_0)\,\mathrm{d}z + \int_{T^{(n)}} \psi(z)(z-z_0),$$

where the first two integrals vanish since the polynomials 1 and $z - z_0$ are entire (and $T^{(n)}$ is closed). So we are left with

$$\int_{T^{(n)}} f(z)\,\mathrm{d}z = \int_{T^{(n)}} \psi(z)(z-z_0).$$

By the ML inequality, we have that

$$\int_{T^{(n)}} f(z)\,\mathrm{d}z \le \sup_{z\in T^{(n)}} |\psi(z)(z-z_0)| \times p^{(n)}.$$

Note that $|z - z_0| \le d^{(n)}$, obviously, and so we have

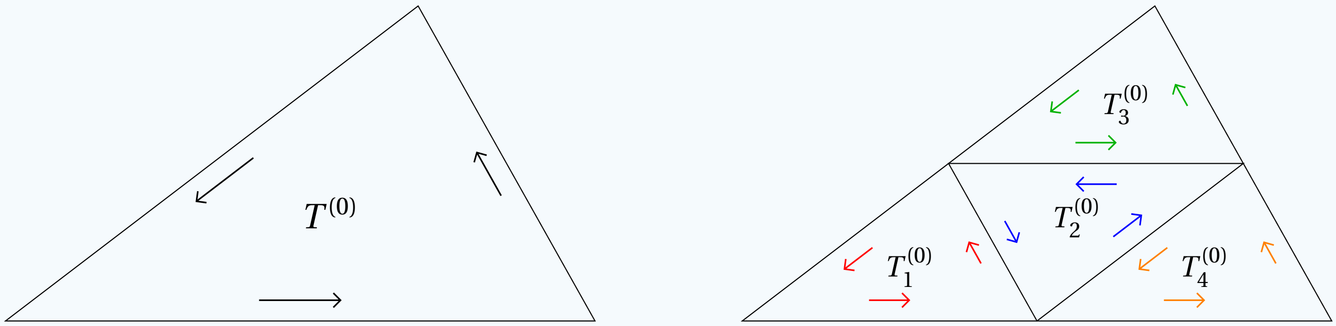
$$\int_{T^{(n)}} f(z)\,\mathrm{d}z \le \varepsilon_n d^{(n)} p^{(n)} = \varepsilon_n d^{(0)} p^{(0)}/4^n,$$

where $\varepsilon_n = \sup_{z\in T^{(n)}} |\psi(z)|$, and since $z \to z_0$ as $n \to \infty$, we have $\varepsilon_n \to 0$ as $n \to \infty$. Finally, we have that

$$\left|\int_{T^{(0)}} f(z)\,\mathrm{d}z\right| \le 4^n\left|\int_{T^{(n)}} f(z)\,\mathrm{d}z\right| \le \varepsilon_n d^{(0)} p^{(0)} \to 0,$$

and so $\int_T f(z)\,\mathrm{d}z = 0$. ❦

[2.0.2] Remark (*Figure for Goursat's Theorem*)



[2.0.3] Corollary

Suppose f is holomorphic on an open set Ω. Let $R \subseteq \Omega$ be some path that traces out a rectangle whose interior also lies in Ω. Then

$$\int_R f(z)\,\mathrm{d}z = 0.$$

> **Proof.** Split the rectangle about the diagonal, yielding two triangles, and apply Goursat's theorem to each. ❦

2.1 Cauchy's Theorem in a Disk

[2.1.1] Lemma (*Holomorphic Function on Open Disk Admits Primitive*)

Suppose f is holomorphic on the open disk Ω. Then f admits a primitive.

Proof. Without loss of generality, let Ω be centered at the origin. Let $z \in \Omega$ be arbitrary, and consider a path $0 \to \mathrm{Re}(z) \to z$ in the orientation specified, and let it be γ_z.

Then define F by

$$F(z) = \int_{\gamma_z} f(w)\,\mathrm{d}w.$$

We will prove that F is holomorphic on Ω, and that $F'(z) = f(z)$. Fixing $z \in \Omega$, let $h \in \mathbb{C}$ be such that $z + h \in \Omega$ (by virtue of Ω being an open disk). Then write

$$F(z+h) - F(z) = \int_{\gamma_{z+h}} f(w)\,\mathrm{d}w - \int_{\gamma_z} f(w)\,\mathrm{d}w.$$

We may draw out the contours γ_{z+h} and $-\gamma_z$, cancel out the contour from $0 \to \mathrm{Re}(z)$, then draw an auxiliary triangle and yield the resultant contour γ_{res} as a straight line connecting $z, z+h$.

Thus

$$F(z+h) - F(z) = \int_{\gamma_{\text{res}}} f(w)\,\mathrm{d}w.$$

Note that f is continuous at z, so we may write

$$f(w) - f(z) = \psi(w) \implies f(w) = f(z) + \psi(w),$$

where $\psi(z) \to 0$ as $w \to z$. Thus we may write the integral as

$$F(z+h) - F(z) = f(z)\int_{\gamma_{\text{res}}} \mathrm{d}w + \int_{\gamma_{\text{res}}} \psi(w)\,\mathrm{d}w.$$

Obviously, the first integral has the primitive w, and so the first term evaluates to $f(z) \times h$. As for the second integral, we may use the ML inequality to yield

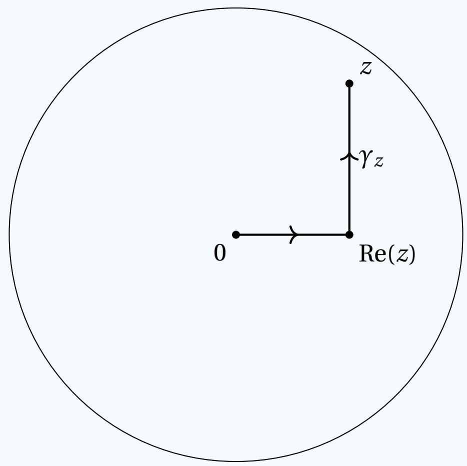
$$\left|\int_{\gamma_{\text{res}}} \psi(w)\,\mathrm{d}w\right| \le \sup_{w \in \gamma_{\text{res}}} |\psi(w)| \times |h|.$$

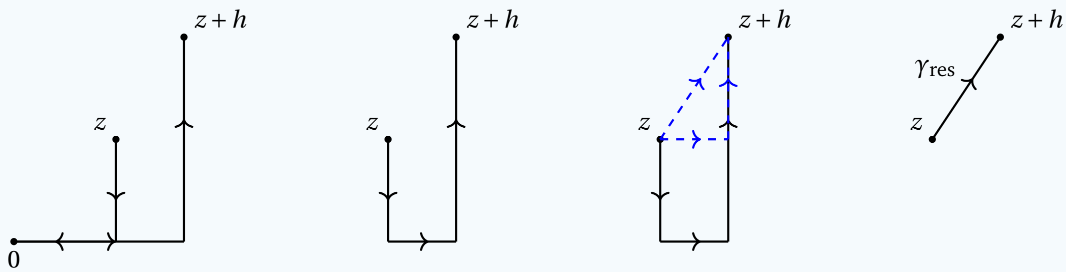
As $h \to 0$, note that supremum goes to 0, and so we have

$$\lim_{h \to 0} \frac{F(z+h) - F(z)}{h} = F'(z) = f(z)$$

by simple rearrangement, completing the proof. ❦

[2.1.2] Remark (*Figures for Cauchy Theorem in a Disk*)





(a) Combined paths (b) After cancellation (c) Auxiliary contours (d) γ_{res}

[2.1.3] Theorem (*Cauchy's Theorem on a Disk*)

Suppose f is holomorphic on some open disk Ω. Then, for any closed curve $\gamma \subseteq \Omega$, we have that

$$\int_\gamma f(z)\,\mathrm{d}z = 0.$$

Proof. f admits a primitive on Ω. ❦

The idea that a holomorphic function on a disk attains a primitive is not unique to disks only. Indeed, the technique of defining our primitive by integrating over a piecewise-smooth path relies purely on our ability to systematically construct polygonal paths, consisting of finitely many horizontal and vertical segments, from a fixed base point z_0 to any point z in the domain without exiting the region of holomorphicity. This geometric framework generalizes naturally to the class of **toy contours**—such as keyholes, sectors, and polygons—where the definition of the "interior" remains intuitive. Within these toy domains, Goursat's theorem for rectangles and triangles guarantees that integrals along any two such polygonal paths agree, yielding a well-defined, single-valued primitive

$$F(z) = \int_{\gamma_z} f(w)\,\mathrm{d}w,$$

which immediately forces the integral over the closed boundary contour to vanish.

However, extending this primitive construction to arbitrary simple closed curves in the plane introduces deep topological difficulties. Defining the interior of a general closed curve rigorously is a non-trivial task, as the curve itself may be highly non-convex or geometrically intricate. Because of the complexity inherent in establishing that the interior of any simple closed curve is well-defined and simply connected, the general proof of Cauchy's theorem for all piecewise-smooth Jordan curves is deferred to Appendix B. There, we will use the topological properties of winding numbers and the Jordan Curve Theorem to establish the equivalence between holomorphic and topological simple connectivity, rigorously completing the generalization of Cauchy's theorem.