2 Topological Spaces and Continuous Functions

2.1 Topological Spaces

[2.1.1] Definition (*Topology, Topological Space*)

A **topology** on a set X is a collection $\mathcal{T}$ of subsets of X with the following properties.

(1) $\varnothing$ and X are elements of $\mathcal{T}$.

(2) The union of the elements of any subcollection of $\mathcal{T}$ is an element of $\mathcal{T}$.

(3) The intersection of the elements of any finite subcollection of $\mathcal{T}$ is an element of $\mathcal{T}$.

A set X endowed with some topology $\mathcal{T}$ is said to be a **topological space**.

[2.1.2] Definition (*Open Set*)

If $(X, \mathcal{T})$ is some topological space, we say $U \subseteq X$ is **open** if $U \in \mathcal{T}$.

[2.1.3] Remark (*Reformulation of Topological Space*)

With this, we may reformulate a topological space to be a set X together with some collection of subsets, called open sets, such that $\varnothing$ and X are open, as well as the arbitrary union of open sets being open, and the finite intersection of open sets being open.

[2.1.4] Example (*Discrete, Indiscrete/Trivial Topology*)

For any set X, the **discrete topology** is said to be the collection of every single subset of X. The **indiscrete/trivial topology** is said to be the collection containing only $\varnothing$ and X.

[2.1.5] Example (*Finite Complement Topology*)

Fix a set X. Define a topology $\mathcal{T}_f$ such that $U \subseteq X$ is in the topology if $X \setminus U$ is either finite or X. Immediately we see $\varnothing$ and X are elements of the topology, so now we must show the latter two axioms hold true.

Let $\{U_\lambda\}_{\lambda \in \Lambda}$ be an indexed family of open sets. Then note

$$X \setminus \bigcup_{\lambda \in \Lambda} U_\lambda = \bigcap_{\lambda \in \Lambda} (X \setminus U_\lambda).$$

Since each U_λ is open, it follows that the left-hand side is an arbitrary intersection of finite sets, which is finite. Thus the arbitrary union of open sets is open.

Let $U_1, \dots, U_n$ be finitely many open sets. Then note

$$X \setminus \bigcap_{i=1}^{n} U_i = \bigcup_{i=1}^{n} (X \setminus U_i).$$

Since each U_i is open, it follows that the left-hand side is a finite union of finite sets, which is finite. Thus the finite intersection of open sets is open. Thus $\mathcal{T}_f$ is a well-defined topology.

[2.1.6] Definition (*Finer, Coarser, and Comparable Topologies*)

Suppose X is a set with two topologies, $\mathcal{T}$ and $\mathcal{T}'$. If $\mathcal{T} \subseteq T'$, then $\mathcal{T}$ is **coarser** than $\mathcal{T}'$. If $\mathcal{T} \neq \mathcal{T}'$, then $\mathcal{T}$ is **strictly coarser** than $\mathcal{T}'$. Conversely, $\mathcal{T}'$ is **finer** or **strictly finer** than $\mathcal{T}$. If $\mathcal{T} \subseteq \mathcal{T}'$ or $\mathcal{T}' \subseteq T$, then we say the two topologies are **comparable**.

2.2 Basis for a Topology

[2.2.1] Definition (*Basis, Basis Elements*)

If X is a set, then a **basis** for some topology on X is a collection $\mathcal{B}$ of subsets of X (called **basis elements**) that satisfy the following conditions.

(1) **Covering.** For each $x \in X$, there is some $B \in \mathcal{B}$ such that $x \in B$. That is, $X \subseteq \bigcup_{B \in \mathcal{B}} B$.

(2) **Refinement.** If there exist $B_1, B_2 \in \mathcal{B}$ such that $x \in B_1 \cap B_2$, there is some $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

[2.2.2] Definition (*Topology Generated by Basis*)

If there exists some basis $\mathcal{B}$ for X, we say the topology $\mathcal{T}$ **generated by** $\mathcal{B}$ is as follows:

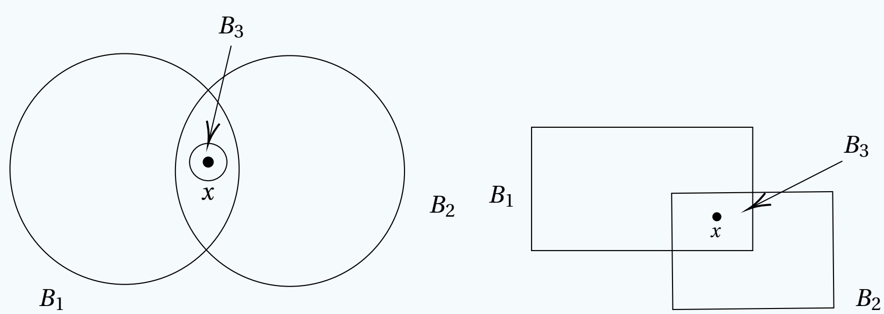
"A subset $U \subseteq X$ is said to be open (i.e., in $\mathcal{T}$) if, for each $x \in U$, there is some $B \in \mathcal{B}$ such that $x \in B \subseteq U$".

[2.2.3] Example (*Basis of Open Balls/Boxes*)

Let $\mathcal{B}$ be the collection of all open balls in $\mathbb{R}^n$. Of course, each $x \in \mathbb{R}^n$ is contained in some open ball $B \in \mathcal{B}$. Now consider some $x \in \mathbb{R}^n$ contained in two open balls $B_1, B_2 \in \mathcal{B}$—that is, $x \in B_1 \cap B_2$. Then we can construct a smaller open ball about x, say B_3, such that $x \in B_3 \subseteq B_1 \cap B_2$ (with $B_3 \in \mathcal{B}$), of course.

We can do an analogous procedure, but instead defining $\mathcal{B}$ to be the collection of all open boxes in $\mathbb{R}^n$. But note that intersections of boxes are, well, boxes, so we do not need to find a basis element inside the intersection of two other basis elements as done with the basis of open balls.

We provide a schematic for these bases in $\mathbb{R}^2$.



[2.2.4] Example (*Basis that Generates Discrete Topology*)

If X is any set, then we can define a basis $\mathcal{B}$ comprising every singleton subset of X. Let $U \subseteq X$ be any arbitrary subset of X. Then note, for each $x \in U$, it's true that $\{x\} \in \mathcal{B}$, such that $x \in \{x\} \subseteq U$. Thus all subsets of X are open, meaning the topology generated by $\mathcal{B}$ is in fact the discrete one.

We now verify that a topology generated by a basis is, indeed, a topology.

[2.2.5] Theorem (*Basis Generates a Topology*)

Suppose $\mathcal{B}$ is a basis for some set X. Let $\mathcal{T}$ be a collection of subsets such that, if some subset $U \subseteq X$ is in $\mathcal{T}$, then for each $x \in U$, there is some $B \in \mathcal{B}$ such that $x \in B \subseteq U$. Then $(X, \mathcal{T})$ forms a topological space.

Proof. First, note $\emptyset \in \mathcal{T}$ vacuously. Next, note that for each $x \in X$, there is some $B_x \in \mathcal{B}$ such that $x \in B_x \subseteq X$ (both by definition of a basis element), so $X \in \mathcal{T}$.

We show arbitrary union is closed in $\mathcal{T}$. Suppose $\{U_\lambda\}_{\lambda \in \Lambda}$ is an indexed family of elements of $\mathcal{T}$. Define

$$U = \bigcup_{\lambda \in \Lambda} U_\lambda.$$

If $x \in U$, then there is some $\lambda \in \Lambda$ such that $x \in U_\lambda$. Since U_λ is open, there exists some $B_\lambda \in \mathcal{B}$ such that $x \in B_\lambda \subseteq U_\lambda \subseteq U$. Thus U is open.

Finally, we show finite intersection is closed in $\mathcal{T}$. That is, for some $n \in \mathbb{Z}_+$, the intersection of open sets $U_1 \cap \cdots U_n$ is open. We prove this inductively, using $n = 2$ as our base case. Suppose $U_1, U_2 \in \mathcal{T}$—we show $U_1 \cap U_2 \in \mathcal{T}$ as well. First, suppose $x \in U_1 \cap U_2$. Then there exist $B_1, B_2 \in \mathcal{T}$ such that $x \in B_1 \subseteq U_1$ and $x \in B_2 \subseteq U_2$. Since $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2 \subseteq U_1 \cap U_2$. Thus $U_1 \cap U_2 \in \mathcal{T}$.

Now suppose this result is true for $n - 1$. Then note

$$U_1 \cap \cdots \cap U_n = \underbrace{(U_1 \cap \cdots U_{n-1})}_{:=U_1} \cap \underbrace{U_n}_{:=U_2}.$$

U_1 is open by hypothesis, of course. Since we have reduced this to $n = 2$, it follows that the intersection is, indeed, finite. Thus $\mathcal{T}$ is a topology! ❦

[2.2.6] Corollary (*Intuition Behind Basis*)

Let X be a set, and let $\mathcal{B}$ be a basis for some topology $\mathcal{T}$ on X. Then $\mathcal{T}$ is the collection of all unions of elements of $\mathcal{B}$.

Proof. Let $\{B_\lambda\}_{\lambda \in \Lambda}$ be a collection of elements of $\mathcal{B}$. Of course, each basis element is an element of the topology it generates. Since topologies are closed under arbitrary union, the union of elements in this collection is a subset of $\mathcal{T}$.

Conversely, suppose $T \in \mathcal{T}$. For each $x \in T$, there is some $B_x \in \mathcal{B}$ such that $x \in B_x \subseteq T$. Thus $T \subseteq \bigcup_{x \in T} B_x$, a union of a collection of elements from $\mathcal{B}$, completing the proof. ❦

Immediately, after choosing a basis for a topology, we note that every element of the topology may be generated by a union of some elements of the basis. This is reminiscent of vectors being generated by a linear combination of some elements of the basis. But note that each vector is uniquely expressible in a vector space's basis, while each element of a topology does not have a necessarily unique expression in the topology's basis.

[2.2.7] Theorem

Suppose $(X, \mathcal{T})$ is a topological space. Let $\mathcal{C}$ be a collection of open sets such that, for each open set $U \subseteq X$ (under the topology $\mathcal{T}$) and for each $x \in U$, there is some $C \in \mathcal{C}$ such that $x \in C \subseteq U$. Then $\mathcal{C}$ is a basis for the topology $\mathcal{T}$.

Proof. First, we show $\mathcal{C}$ is a basis. Note X is open, so by hypothesis, every $x \in X$ is met with some $C \in \mathcal{C}$ such that $x \in C$, and of course $C \subseteq X$. Moreover, suppose $x \in X$ is such that, for some $C_1, C_2 \in \mathcal{C}$, it follows that $x \in C_1 \cap C_2$. Since C_1, C_2 are open under $\mathcal{T}$, it follows that $C_1 \cap C_2$ are open under $\mathcal{T}$, so there exists some $C_3 \in \mathcal{C}$ such that $x \in C_3 \subseteq C_1 \cap C_2$ by hypothesis. Thus $\mathcal{C}$ is a basis.

Now we show the topology generated by $\mathcal{C}$, say $\mathcal{T}'$, is $\mathcal{T}$ itself. This is slightly tautological, but whatever. Suppose $U \subseteq X$ is open under $\mathcal{T}$. The, for each $x \in U$, there is some $C \in \mathcal{C}$ such that $x \in C \subseteq U$ (by hypothesis), meaning $U \in \mathcal{T}'$. Now suppose $W \in \mathcal{T}'$. Then W is expressible as the arbitrary union of elements of $\mathcal{C}$, which are all elements of $\mathcal{T}$, and so $W \in \mathcal{T}$. Thus $\mathcal{T} = \mathcal{T}'$. ❦

It is difficult to describe topologies by their elements (as a topology is a subset of a power set, which is unwieldy). Thus we look towards a topological space's basis, a typically smaller subset of the topology that generates it under unions. We investigate properties of topologies by looking at bases of a topology.

[2.2.8] Theorem

Suppose X is some set with topologies $\mathcal{T}, \mathcal{T}'$, each with a basis $\mathcal{B}, \mathcal{B}'$ respectively. Then the following statements are equivalent.

(1) $\mathcal{T}$ is coarser than $\mathcal{T}'$.

(2) For each $x \in X$ and for every $B \in \mathcal{B}$ such that $x \in B$, there exists some $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$.

Proof.

(2) $\Longrightarrow$ (1): We seek to show $\mathcal{T} \subseteq \mathcal{T}'$. Suppose $U \in \mathcal{T}$. Then, since $\mathcal{B}$ generates $\mathcal{T}$, there is some $B \in \mathcal{B}$ such that $x \in \mathcal{B} \subseteq U$. By hypothesis, however, there is some $B' \in \mathcal{T}'$ such that $x \in \mathcal{B}' \subseteq U$. Thus $\mathcal{B}'$ generates $\mathcal{T}$, meaning $U \subseteq \mathcal{T}'$.

(1) $\Longrightarrow$ (2): Suppose $x \in X$ is such that there is some $B \in \mathcal{B}$ for which $x \in B$. Note $\mathcal{B} \subseteq \mathcal{T} \subseteq \mathcal{T}'$, and so $B \in \mathcal{T}'$. Since $\mathcal{T}'$ is generated by $\mathcal{B}'$, there is some $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$. ❦

[2.2.9] Example (*Topologies on* $\mathbb{R}$)

Consider the following topoologies that can be endowed on $\mathbb{R}$.

(1) Let $\mathcal{B}$ be the set of all open intervals in $\mathbb{R}$.

$$\mathcal{B} = \{(a,b) : a < b\} = \{x \in \mathbb{R} : a < x < b\}.$$

The topology generated by $\mathcal{B}$ is said to be the **standard topology** on $\mathbb{R}$. All open sets of $\mathbb{R}$ with respect to this topology take the form of the union of arbitrarily many open intervals.

(2) Now let $\mathcal{B}$ be defined as such.

$$\mathcal{B} = \{[a,b) : a < b\} = \{x \in \mathbb{R} : a \le x < b\}.$$

The topology generated by $\mathcal{B}$ is said to be the **lower limit topology**. We denote $\mathbb{R}$ endowed with this topology as $\mathbb{R}_\ell$.

(3) Finally, define $K = \{1/n\}_{n \in \mathbb{Z}_+}$, and let $\mathcal{B}$ be defined as such.

$$\mathcal{B} = \{(a,b), (a,b) \setminus K\}.$$

The topology generated by $\mathcal{B}$ is said to be the **K-topology**. We denote $\mathbb{R}$ endowed with this topology as $\mathbb{R}_K$.

[2.2.10] Remark

The topologies on $\mathbb{R}_\ell$ and $\mathbb{R}_K$ (denoted $\mathcal{T}_\ell$ and $\mathcal{T}_K$) are both strictly finer than the topology on $\mathbb{R}$ (denoted $\mathcal{T}$) and both incomparable.

Proof. First, we show $\mathcal{T}_\ell$ is strictly finer than $\mathcal{T}$. Let $x \in (a,b)$, with (a,b) an arbitrary basis element of $\mathcal{T}$. Then note $x \in [x,b) \subseteq (a,b)$, where $[x,b)$ is a basis element of $\mathcal{T}_\ell$, and so $\mathcal{T} \subseteq \mathcal{T}_\ell$ ($\mathcal{T}_\ell$ is finer than $\mathcal{T}$). To show that this is a strict subset, we show $\mathcal{T}_\ell \not\subseteq \mathcal{T}$. Consider some $x \in [x,d)$, with $[x,d)$ a basis element of $\mathcal{T}_\ell$. Then note there is no open interval that contains x and is a subset of $[x,d)$. Thus $\mathcal{T}$ is not finer than $\mathcal{T}_\ell$, meaning $\mathcal{T}_\ell$ is strictly finer than $\mathcal{T}$.

Next, we show $\mathcal{T}_K$ is strictly finer than $\mathcal{T}$. Note the basis of $\mathcal{T}_K$ comprises the basis of $\mathcal{T}$, so it is automatically finer than $\mathcal{T}$—we seek to show $\mathcal{T}$ is not finer than $\mathcal{T}_K$. Let $0 \in (-1,1) \setminus K$, where $(-1,1) \setminus K$ is a basis element of $\mathcal{T}_K$. Then note there is no local refinement in $\mathcal{T}$ that contains 0 (that is, there is no open interval that contains 0 and is a subset of this basis element). Thus $\mathcal{T}$ is not finer than $\mathcal{T}_K$, meaning $\mathcal{T}_K$ is indeed strictly finer than $\mathcal{T}$.

Finally, we show $\mathcal{T}_\ell$ and $\mathcal{T}_K$ are incomparable (i.e., neither is finer than one another). Consider $2 \in [2,3)$, where $[2,3)$ is a basis element of $\mathcal{T}_\ell$. Then note no open interval (a,b)—much less any punctured open interval $(a,b) \setminus K$—contains 2 and is a subset of $[2,3)$. Thus $\mathcal{T}_K$ is not finer than $\mathcal{T}_\ell$. Conversely, consider $0 \in (-1,1) \setminus K$, with $(-1,1) \setminus K$ a basis element of $\mathcal{T}_K$. No left closed, right open interval (i.e., basis elements of $\mathcal{T}_\ell$) contains 0 and is a subset of $(-1,1) \setminus K$. Thus $\mathcal{T}_\ell$ is not finer than $\mathcal{T}_K$, completing the proof. ❦

Consider some arbitrary topological space $(X, \mathcal{T})$, and let $\mathcal{B}$ be a basis for $\mathcal{T}$. Recall that $\mathcal{B}$ must be a cover for X, but it also carries the extra requirement that the intersection of two basis elements must comprise another basis element. In other words, a basis is closed under finite intersection. Then, what is the minimal structure that, when considering the set of all finite intersections, regenerates the basis? We define the subbasis and explain the hierarchy accordingly.

[2.2.11] Definition (*Subbasis*)

A **subbasis** $\mathcal{S}$ for a topology on X is a collection of subsets of X whose union equals X. The topology generated by the subbasis $\mathcal{S}$ is said to be the collection $\mathcal{T}$ of all unions of fintie intersections of elements of $\mathcal{S}$.

[2.2.12] Remark (*Interlude on Subbasis*)

Before we show the topology generated by a subbasis is, indeed, a topology, we provide some form of intuition for it.

A subbasis $\mathcal{S}$ is the most general collection of subsets capable of generating a topology. The primary distinction between a basis and a subbasis lies in the intersection axiom: while a basis $\mathcal{B}$ must be "self-refining"—in the sense that the intersection of any two basis elements must be representable as a union of other basis elements—a subbasis $\mathcal{S}$ is subject to no such constraint.

To bridge this gap, the construction of the topology $\mathcal{T}$ proceeds in two distinct stages:

(1) **Generating the Basis:** We first take the collection of all finite intersections of elements in $\mathcal{S}$. This produces a new collection $\mathcal{B}$ which is, by construction, a basis. Because $\mathcal{B}$ is formed under finite intersections, it satisfies the "self-refining" property that $\mathcal{S}$ lacked.

(2) **Generating the Topology:** We then take the collection of all arbitrary unions of elements in $\mathcal{B}$. This final collection $\mathcal{T}$ now satisfies the remaining axioms of a topology.

Conceptually, this identifies the topology $\mathcal{T}$ as the collection of all arbitrary unions of finite intersections of elements in $\mathcal{S}$. This is the coarsest/minimal topology containing $\mathcal{S}$, allowing one to define a topology by simply declaring a specific collection of sets to be open, bypassing the need to verify that the initial collection satisfies any intersection properties itself.

[2.2.13] Theorem (*Subbasis Generates a Topology*)

Let X be an arbitrary set. The set of all arbitrary unions of the set of all finite intersections of a subbasis forms a topology on X.

Proof. Let X be a set we seek to endow a topology on. We show that the set of all finite intersections of a subbasis forms a basis, and the theorem follows immediately (since we know a basis generates a topology by taking arbitrary unions). Suppose our subbasis $\mathcal{S}$ is written as

$$\mathcal{S} = \{S_\lambda\}_{\lambda \in \Lambda}.$$

We seek to show $\mathcal{B}$, the set of all finite intersections of elements of $\mathcal{S}$, forms a basis. Of course, $\mathcal{S} \subseteq \mathcal{B}$, and $\mathcal{S}$ covers X, so automatically $\mathcal{B}$ covers X as well. Now let

$$B_1 = S_{\lambda_1} \cap \cdots \cap S_{\lambda_m} \qquad B_2 = S'_{\lambda_1} \cap \cdots \cap S'_{\lambda_n},$$

where $B_1, B_2 \in \mathcal{B}$. To complete the proof, we show some finite intersection of S-sets is a subset of $B_1 \cap B_2$. Indeed,

$$B_3 = B_1 \cap B_2 = (S_{\lambda_1} \cap \cdots \cap S_{\lambda_m}) \cap (S'_{\lambda_1} \cap \cdots \cap S'_{\lambda_n})$$

is a finite intersection of sets (by basic properties of sets), so $B_3 \in \mathcal{B}$. Thus, each $x \in (B_1 \cap B_2)$ is such that $x \in B_3 \subseteq (B_1 \cap B_2)$, completing the proof. ❦

2.2.1 Exercises

[2.2.14] Problem

Let $(X,\mathcal{T})$ be a topological space, with $A\subseteq X$. Suppose, for each $x\in A$, there is some open set U_x such that $x\in U_x\subseteq A$. Show A is open.

Proof. Define $O=\bigcup_{x\in A}U_x$. Note that it is an arbitrary union of open sets, and so it is open. Immediately, since each $U_x\subseteq A$, it follows that $O\subseteq A$. But also, since $\{U_x\}_{x\in A}$ forms a cover for A, it's true that $A\subseteq O$. Thus $A=O$, and so A is open. ❦

[2.2.15] Problem

Suppose $\{\mathcal{T}_\alpha\}_{\alpha\in A}$ is a family of topologies on X. Show $\bigcap_{\alpha\in A}\mathcal{T}_\alpha$ is a topology.

Proof. Let $T=\bigcap_{\alpha\in A}\mathcal{T}_\alpha$. Then T is the set of all sets that were open in every topology in $\{T_\alpha\}_{\alpha\in A}$. We note that $\emptyset$ and X are open in each T_α ($\alpha\in A$), and so they belong in T. Now consider arbitrarily many elements $\{O_\lambda\}_{\lambda\in\Lambda}$ from T. We note these elements are in every $\mathcal{T}_\alpha:\alpha\in A$, so their union should also be in every $\mathcal{T}_\alpha:\alpha\in A$. Thus the union is in T as well. An analogous statement may be made about the intersection of finitely many elements, which completes the proof. ❦

[2.2.16] Problem

Suppose $\mathcal{A}$ is the basis for some topology on X. Show that the topology $\mathcal{A}$ generates is the intersection of all topologies on X that contain $\mathcal{A}$.

Proof. Suppose $\mathcal{A}$ generates a topology T. Let $\{\mathcal{T}_\alpha\}_{\alpha\in A}$ be the family of topologies on X such that $\mathcal{A}\subseteq\mathcal{T}_\alpha$ for each $\alpha\in A$. Let $T'=\bigcap_{\alpha\in A}\mathcal{T}_\alpha$—we show $T=T'$. Of course, $T\in\{\mathcal{T}_\alpha\}_{\alpha\in A}$, and so $T'\subseteq T$. Now we seek to show $T\subseteq T'$. Equivalently, we show that for any open set in T, it follows that aforementioned open set is also in $\mathcal{T}_\alpha$ for every $\alpha\in A$. Fix $\alpha\in A$ and $O\in T$. Note $\mathcal{A}$ is a basis for T, and so for each $x\in O$, there is some $B_x\in\mathcal{A}$ such that $x\in B_x\subseteq O$. Thus $O=\bigcup_{x\in O}B_x$. But note $B_x\in\mathcal{A}\subseteq\mathcal{T}_\alpha$ for every $x\in O$, and so the arbitrary union of each B_x is in $\mathcal{T}_\alpha$. Thus $O\in\mathcal{T}_\alpha$, which completes the proof. ❦

[2.2.17] Problem

Let $\mathcal{B}=\{(a,b):a<b\in\mathbb{Q}\}$. Show $\mathcal{B}$ is a basis for the standard topology on $\mathbb{R}$.

Proof. Let $O\subseteq\mathbb{R}$ be open with respect to the standard topology. Fix $x\in O$, and choose $\varepsilon>0$ such that $(x-\varepsilon,x+\varepsilon)\subseteq O$. By the density of $\mathbb{Q}$ in $\mathbb{R}$, there is some $a<b\in\mathbb{Q}$ such that

$$x-\varepsilon<a<x<b<x+\varepsilon.$$

Then note $(a,b)=B\in\mathcal{B}$, and thus $x\in B\subseteq(x-\varepsilon,x+\varepsilon)\subseteq O$. Thus $\mathcal{B}$ is a basis for the standard topology on $\mathbb{R}$. ❦

2.3 Order Topology

If a set is totally ordered, then the ordering actually induces a canonical topology on X called the **order topology**. We investigate its construction and behavior in this section.

First, considering open/closed/half-open intervals in $\mathbb{R}$, we may define them equivalently in an arbitrary totally ordered set, using its order in place of $\mathbb{R}$'s standard ordering. We name them equivalently, suggesting that open intervals in a totally ordered set should be open with respect to the order topology. This is true.

[2.3.1] Definition (*Order Topology*)

Suppose $(X, <)$ is a totally ordered set. Let $\mathcal{B}$ be the collection of all sets of the following forms.

(1) All open intervals $(a, b) \subseteq X$, for every $a, b \in X$.

(2) The half-open interval $[a_0, b) \subseteq X$, for some minimum element (if any) $a_0 \in X$ and any $b \in X$.

(3) The half-open interval $(a, b_0] \subseteq X$, for any $a \in X$ and some maximum element (if any) $b_0 \in X$.

Then $\mathcal{B}$ forms a basis for a topology on X called the **ordered topology**.

Proof. We show $\mathcal{B}$ is, indeed, a basis. First, note that every element of X lies in one of the three prescribed sets: the smallest lies in the upper-open set, the largest lies in the lower-open set, and all other elements lie in an open interval—thus $\mathcal{B}$ covers X as promised. With a long argument by cases, it follows that the intersection of two basis elements can be locally refined. ❦

Note the standard topology on $\mathbb{R}$ is simply the topology induced by the standard order. Note the order topology on $\mathbb{Z}_+$ is simply the discrete topology, since every subset of $\mathbb{Z}_+$ falls into an interval (the singleton $\{n\} = (n-1, n+1)$).

[2.3.2] Definition (*Rays*)

If X is a totally ordered set, an element $a \in X$ determines four rays. They are $(a, \infty), [a, \infty), (-\infty, a), (-\infty, a]$.

[2.3.3] Remark (*Open Rays are Open*)

The rays of form (a, ∞) and $(-\infty, a)$ are called **open rays**. Its name is accurate—this interval forms an open set with respect to the order topology. For a set X whose maximum b_0, $(a, \infty) = (a, b_0]$; if no maximum exists, then (a, ∞) is simply the union of all basis elements (a, x) for $x : x > a$. Similar argumentation follows for $(-\infty, a)$.

Please note that, for a totally ordered set X with maximum (resp. minimum) b_0 (resp. a_0), then $(a, \infty) = (a, b_0]$ (resp. $(-\infty, a) = [a_0, b)$).

[2.3.4] Theorem (*Open Rays Form Subbasis for Order Topology*)

Suppose X is a totally ordered set. Then the set

$$S = \{(-\infty, a) : a \in X\} \cup \{(a, \infty) : a \in X\}$$

forms a subbasis for the order topology on X.

Proof. It should follow immediately that every basis element can be generated by a finite intersection of one or two open rays. ❦

2.4 Product Topology

[2.4.1] Definition (*Product Topology*)

Suppose X and Y are sets endowed with certain topologies $\mathcal{T}_X, \mathcal{T}_Y$. Define $\mathcal{B}$ to be the collection of sets $U \times V$, where U and V are open subsets of X and Y (with respect to $\mathcal{T}_X$ and $\mathcal{T}_Y$, respectively). Then $\mathcal{B}$ generates the **product topology** on $X \times Y$.

Proof. We show $\mathcal{B}$ is, indeed, a basis. Fix $(x, y) \in (X, Y)$. Since topologies cover their respective set, there exist open subsets U, V of X, Y—thus $(x, y) \in U \times V$, meaning $\mathcal{B}$ covers $X \times Y$. Now let $U_1, U_2 \subseteq X$ and $V_1, V_2 \subseteq Y$ be open. Then

$$(U_1 \times V_1) \cap (U_2 \times V_2) = \underbrace{(U_1 \cap U_2)}_{\in \mathcal{T}_X} \times \underbrace{(V_1 \times V_2)}_{\in \mathcal{T}_Y}.$$

Since the intersection of two open sets is open, we have that the intersection of two basis elements is, indeed, another basis element. ❦

[2.4.2] Theorem (*Product of Bases Generates Product Topology*)

Suppose $\mathcal{B}_X$ and $\mathcal{B}_X$ are bases for topologies on sets X and Y. Then the collection of sets

$$\mathcal{B} = \{B_X \times B_Y : B_X \in \mathcal{B}_X, B_y \in \mathcal{B}_Y\}$$

forms a basis for the product topology of $X \times Y$.

Proof. First, we show $\mathcal{B}$ covers $X \times Y$. Fix $(x, y) \in X \times Y$. Then note $\mathcal{B}_X$ and $\mathcal{B}_Y$ cover X and Y, so there exists $B_X \in \mathcal{B}_X$ and $B_Y \in \mathcal{B}_Y$ such that $(x, y) \in B_X \times B_Y$—thus $\mathcal{B}$ covers $X \times Y$.

Now let $(x, y) \in U \times V \subseteq X \times Y$, where U and V are open subsets of X and Y, respectively. Of course, $U \times V$ is open with respect to the product topology, so we simply have to locally refine it for the specific point (x, y). Note that there is some $B_X \in \mathcal{B}_X$ (resp. $B_Y \in \mathcal{B}_Y$) such that $x \in B_X \subseteq U$ (resp. $y \in B_Y \subseteq V$), and so $(x, y) \in (B_X \times B_Y) \subseteq (U \times V)$. Thus elements of an open subset of $X \times Y$ can be locally refined by $\mathcal{B}$. Thus it is a basis, completing the proof. ❦

[2.4.3] Remark (*Basis for* $\mathbb{R}^2$)

Recall the standard topology on $\mathbb{R}$ is the order topology. Then, in endowing $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$ with the product topology, we seek a basis for $\mathbb{R}^2$. By definition, the basis is simply the product of all open subsets of $\mathbb{R}$. But note, with our previous theorem, we may in fact reduce the basis to simply all sets of form $(a, b) \times (c, d)$, noting that the set of all open intervals of $\mathbb{R}$ forms a basis for the standard topology on $\mathbb{R}$.

[2.4.4] Definition (*Projection Maps*)

Suppose X and Y are sets. Define the following projection maps

$$\pi_X : X \times Y \to X \quad \pi_X(x, y) = x,$$
$$\pi_Y : X \times Y \to Y \quad \pi_Y(x, y) = y.$$

[2.4.5] Theorem (*Subbasis for Product Topology*)

The collection

$$\mathcal{S} = \{\pi_X^{-1}(U) : U \text{ open in } X\} \cup \{\pi_Y^{-1}(V) : V \text{ open in } Y\}$$

forms a subbasis for the product topology on $X \times Y$.

Proof. Note that $\pi_X^{-1}(U) = U \times X$ and $\pi_Y^{-1}(V) = V \times Y$. For any two open subsets $U \subseteq X$ and $V \subseteq Y$, we have that

$$\pi_X^{-1}(U) \cap \pi_Y^{-1}(V) = (U \times Y) \cap (V \times X) = (U \times V) \cap (X \times Y) = U \times V.$$

Thus every element of the standard basis for the product topology on $X \times Y$ is generated by the subbasis. Moreover, since U, V are arbitrary, these are the only types of elements produced by the finite intersection of subbasis elements. Thus the intersection of all subbasis elements produces the standard basis for the product topology. ❦

2.5 Subspace Topology

[2.5.1] Definition

Let $(X, \mathcal{T}_X)$ be a topological space, and let $Y \subseteq X$. Then the collection

$$\mathcal{T}_Y = \{Y \cap U : U \in \mathcal{T}_X\}$$

is the **subspace topology** on Y.

Proof. Note that $\emptyset, X \in \mathcal{T}_X$, and these sets in $\mathcal{T}_X$ correspond directly to $\emptyset, Y$ in $\mathcal{T}_Y$. Now suppose $\{Y_\lambda\}_{\lambda\in\Lambda}$ is a collection of sets in $\mathcal{T}_Y$, where each $Y_\lambda = Y \cap U_\lambda$. Then

$$\bigcup_{\lambda\in\Lambda} Y_\lambda = \bigcup_{\lambda\in\Lambda} Y \cap U_\lambda = Y \cap \underbrace{\left(\bigcup_{\lambda\in\Lambda} U_\lambda\right)}_{\in \mathcal{T}_X},$$

and so $\mathcal{T}_Y$ is closed under arbitrary union. An entirely symmetric argument follows to show $\mathcal{T}_Y$ is closed under finite intersection—thus $\mathcal{T}_Y$ is a topology. ❦

[2.5.2] Theorem

Suppose $(X, \mathcal{T}_X)$ is a topological space with basis $\mathcal{B}$. Then, for any subspace $(Y, \mathcal{T}_Y)$, the collection

$$\mathcal{C} = \{B \cap Y : B \in \mathcal{B}\}$$

forms a basis for $\mathcal{T}_Y$.

Proof. Let $y \in Y$. Then note some $O \in \mathcal{T}_X$ is such that $y \in O \subseteq X$. But then $O \cap Y$ is a basis element, so $y \in (O \cap Y) \subseteq O \subseteq X$. Thus $\mathcal{C}$ covers Y and locally refines every element, making it a basis. ❦

Note that if X is a topological space and $Y \subseteq X$ is endowed with the subspace topology, then any open subset in Y is simply the intersection of some open set in X with Y.

[2.5.3] Corollary (*Subspace Topology is a Subset of Parent Topology for Open Subspaces*)

Suppose $(Y, \mathcal{T}_Y)$ is an **open** subspace of $(X, \mathcal{T}_X)$. Then $\mathcal{T}_Y \subseteq \mathcal{T}_X$. Equivalently, all open sets in Y are open in X.

Proof. Suppose $O \in \mathcal{T}_Y$. Then O is an arbitrary union of basis elements of $\mathcal{T}_Y$. Let $\mathcal{B}$ be a basis for $\mathcal{T}_X$. Then we may write O as the arbitrary union of basis elements, specifically with the basis induced by $\mathcal{B}$. Then

$$O = \bigcup_{\lambda\in\Lambda} Y \cap U_\lambda = \underbrace{Y}_{\in \mathcal{T}_X} \cap \underbrace{\left(\bigcup_{\lambda\in\Lambda} U_\lambda\right)}_{\in \mathcal{T}_X} \in \mathcal{T}_X,$$

where $\{U_\lambda\}_{\lambda\in\Lambda}$ is a set of open sets in X. Then the arbitrary union of each U_λ is open in X, and since Y is open as well, the intersection of this arbitrary union with Y is open as well. Thus O is open in X, meaning $\mathcal{T}_Y \subseteq \mathcal{T}_X$. ❦

[2.5.4] Theorem (*Product Topology and Subspace Topology Coincide*)

Suppose X, Y are topological spaces with subspaces $U \subseteq X$ and $V \subseteq Y$, endowed with the subspace topology. Then the subspace topology on $U \times V \subseteq X \times Y$ is exactly the product topology.

Proof. Let $\mathcal{T}_X, \mathcal{T}_Y, \mathcal{T}_U, \mathcal{T}_V$ be topologies endowed on the respective sets. Let $\mathcal{T}_S$ be the subspace topology for $U \times V \subseteq X \times Y$, and let $\mathcal{T}_{U\times V}$ be the product topology on $U \times V$. Now define these collections of sets as bases for the topologies:

$$\mathcal{B}_S = \{(O_X \times O_Y) \cap (U \times V) : O_X \in \mathcal{T}_X, O_Y \in \mathcal{T}_Y\},$$

$$\mathcal{B}_{U\times V} = \{(O_U \times O_V) : O_U \in \mathcal{T}_U, O_V \in \mathcal{T}_V\}.$$

Let $O_S \in \mathcal{T}_S$. Then note O_S is the arbitrary union of $\{B_{S_\lambda}\}_{\lambda\in\Lambda}$, and so

$$O_S = \bigcup_{\lambda\in\Lambda} (O_{X_\lambda} \times O_{Y_\lambda}) \cap (U \times V) = \bigcup_{\lambda\in\Lambda} \underbrace{\underbrace{(O_{X_\lambda} \cap U)}_{\in \mathcal{T}_U} \times \underbrace{(O_{Y_\lambda} \cap V)}_{\in \mathcal{T}_V}}_{\in \mathcal{T}_{U\times V}} \in \mathcal{T}_{U\times V}.$$

Thus $\mathcal{T}_S \subseteq \mathcal{T}_{U\times V}$. Conversely, suppose $O_{U\times V} \in \mathcal{T}_{U\times V}$. Then

$$O_{U\times V} = \bigcup_{\lambda\in\Lambda} (O_{U_\lambda} \times O_{V_\lambda}) = \left(\bigcup_{\lambda\in\Lambda} (O_{X_\lambda} \times O_{Y_\lambda})\right) \cap (U \times V) \in \mathcal{T}_S,$$

and so $\mathcal{T}_S \supseteq \mathcal{T}_{U\times V}$, completing the proof. ❦

The product topology is nice in this regard, but note the order topology is not. That is, if $(X, <)$ is an ordered set and $(Y, <)$ is a subspace with the same ordering, the order topology is not necessarily the inherited subspace topology. We provide an example where the subspace and order topology do and don't coincide.

[2.5.5] Example (*Subspace Topology vs. Order Topology*)

For our first example, let $Y_1 = [0,1]$ be a subset of $\mathbb{R}$. Consider its subspace and order topologies. The basis for the subspace topology takes the form

$$\mathcal{B}_S = \{(a,b) \cap Y_1 : a, b \in \mathbb{R}\} = \begin{cases} (a,b) & a, b \in Y_1, \\ [0,b) & b \in Y_1, \\ (a,1] & a \in Y_1, \\ Y \text{ or } \varnothing & \text{otherwise.} \end{cases}$$

Note this coincides exactly with the basis for the order topology (excluding $Y, \varnothing$, but they are redundant when generating the topology). Thus the subspace and order topologies are equivalent.

Now let $Y_2 = [0,1) \cup \{2\}$. The basis for the subspace topology takes the form

$$\mathcal{B}_S = \{(a,b) \cap ([0,1) \cup \{2\}) : a, b \in \mathbb{R}\} = \{[(a,b) \cap [0,1)] \cup [(a,b) \cap \{2\}] : a, b \in \mathbb{R}\}.$$

Namely, note that if we choose $(a,b) = (1.5, 2.5)$, we have that $(a,b) \cap Y = \{2\} \in \mathcal{B}_S$, and so it is a member of the subspace topology. But note that any element of the order topology that contains 2 must either be $[0,1) \cup \{2\}$ or $(a,1) \cup \{2\}$, where $0 < a < 1$. In any case, $\{2\}$ is not in the order topology, meaning the order topology and subspace topology do not agree for Y_2.

[2.5.6] Definition (*Convex Set*)

Suppose $(X, <)$ is an ordered set and $(Y, <) \subseteq (X, <)$. We say Y is **convex** in X if, for any $a < b \in Y$, the open interval $(a,b)_X \subseteq Y$.

For example, $[0,1]$ is a convex subset of $\mathbb{R}$, but $[0,1) \cup \{2\}$ is not (since $(1,2)_{\mathbb{R}}$ contains elements like 1.5).

[2.5.7] Theorem (*Order Topology and Subspace Topology Coincide for Convex Subsets*)

Suppose $(X, <)$ is an topological space endowed with the order topology. Let $(Y, <)$ be a subspace of X, where $Y \subseteq X$ is convex. Then the subspace topology and order topology coincide.

Proof. Without loss of generality, suppose Y has a maximum and minimum. Then a basis element for the subspace topology on Y takes the form

$$x \in (\mathcal{B}_S = \{(a,b) \cap Y : a,b \in X\}) = \begin{cases} (a,b) & a,b \in Y, \\ [\min(Y), b) & b \in Y, \\ (a, \max(Y)] & a \in Y, \\ Y \text{ or } \emptyset & \text{otherwise}, \end{cases}$$

which is exactly the order topology. Of course, note that the second and third forms of basis elements are removed if Y does not have a minimum or maximum, respectively. Nevertheless, this coincides perfectly with the order topology on Y, completing the proof. ❦

2.5.1 Exercises

[2.5.8] Problem

Suppose Y is a subspace of X and $A \subseteq Y$. Show that the subspace topology A inherits from Y is the same as the subspace topology A inherits from X.

Proof. Let $\mathcal{T}_{ST}$ denote the subspace topology endowed on set S as a subspace of T. Let $\mathcal{T}_X$ be the topology on X. We show $\mathcal{T}_{AY} \subseteq \mathcal{T}_{AX} \subseteq \mathcal{T}_{AY}$.

First, suppose $O \in \mathcal{T}_{AY}$. Then note there exist a family of open sets $\{O_{Y_\lambda}\}_{\lambda\in\Lambda} \subseteq \mathcal{T}_{YX}$ such that

$$O = \bigcup_{\lambda\in\Lambda} (O_{Y_\lambda} \cap A) = A \cap \left(\bigcup_{\lambda\in\Lambda} (O_{Y_\lambda}) \right) = A \cap \left(\bigcup_{\lambda\in\Lambda} (O_{X_\lambda} \cap Y) \right) = \overset{A}{\cancel{(A \cap Y)}} \cap \left(\bigcup_{\lambda\in\Lambda} O_{X_\lambda} \right) \in \mathcal{T}_{AX}.$$

Conversely, let $O \in \mathcal{T}_{AX}$. Then note there exist a family of open sets $\{O_{Y_\lambda}\}_{\lambda\in\Lambda} \subseteq \mathcal{T}_X$ such that

$$O = \bigcup_{\lambda\in\Lambda} (O_{X_\lambda} \cap A) = A \cap \left(\bigcup_{\lambda\in\Lambda} (O_{X_\lambda}) \right) = (A \cap Y) \cap \left(\bigcup_{\lambda\in\Lambda} (O_{X_\lambda}) \right) = A \cap \left(\bigcup_{\lambda\in\Lambda} (O_{X_\lambda} \cap Y) \right) \in \mathcal{T}_{AY},$$

completing the proof.

[2.5.9] Corollary

Suppose $f : X \to Y$ is a function of sets, and Λ is an index set. Show that

$$f\left(\bigcup_{\lambda\in\Lambda} A_\lambda \right) = \bigcup_{\lambda\in\Lambda} f(A_\lambda).$$

Proof. Suppose $y \in f\left(\bigcup_{\lambda\in\Lambda} A_\lambda\right)$. Then there is some $\mu \in \Lambda$ such that $x \in A_\mu$ and $y = f(x)$. Then, of course, $y \in f(A_\mu) \subseteq \bigcup_{\lambda\in\Lambda} f(A_\lambda)$. Conversely, suppose $y \in \bigcup_{\lambda\in\Lambda} f(A_\lambda)$. There is some $\mu \in \Lambda$ such that there is some $x \in A_\mu$ such that $y = f(x) \in f(A_\mu) \subseteq f\left(\bigcup_{\lambda\in\Lambda} A_\lambda\right)$, completing the proof.

[2.5.10] Problem

A map $f : X \to Y$ is said to be an **open map** if, for every open set $U \subseteq X$, the image $f(U)$ is open in Y. Show $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ are open maps.

Proof. Let U be an open subset of $X \times Y$. Then note there exists an indexing set Λ such that $U = \bigcup_{\lambda\in\Lambda}(O_{X_\lambda} \times O_{Y_\lambda})$. Then

$$\pi_X(U) = \pi_X\left(\bigcup_{\lambda\in\Lambda} (O_{X_\lambda} \times O_{Y_\lambda}) \right) = \bigcup_{\lambda\in\Lambda} \pi_X(O_{X_\lambda} \times O_{Y_\lambda}) = \bigcup_{\lambda\in\Lambda} O_{X_\lambda},$$

which is open in X. An analogous procedure can be applied for π_Y.

[2.5.11] PROBLEM

Show that

$$\mathcal{B} = \{(a,b)\times(c,d) : a<b, c<d \text{ and } a,b,c,d\in\mathbb{Q}\}$$

is a basis for the standard topology on $\mathbb{R}^2$.

Proof. Let $O \subseteq \mathbb{R}^2$ be open, and let $x \in O$, where $x = (x_1, x_2)$. Since O is the arbitrary union of open boxes, there is some $a' < b', c' < d' \in \mathbb{R}$ such that $x \in [(a',b')\times(c',d')] \subseteq O$. This means $a' < x_1 < b'$ and $c' < x_2 < d'$. By the density of the rationals, there exist $a,b,c,d \in \mathbb{Q}$ such that

$$a' < a < x_1 < b < b',$$
$$c' < c < x_2 < d < d'.$$

Then $x \in [(a,b)\times(c,d)] \subseteq [(a',b')\times(c',d')] \subseteq O$, meaning $\mathcal{B}$ covers $\mathbb{R}^2$ and locally refines each open set, completing the proof.❦

2.6 Closed Sets and Limit Points

[2.6.1] Definition (*Closed Set*)

A subset A of a topological subspace X is said to be **closed** if its complement $X \setminus A$ is open.

[2.6.2] Example (*Examples of Closed Sets*)

Consider a closed interval $[a,b] \subseteq \mathbb{R}$ equipped with the standard topology. Then note $[a,b]^C = (-\infty, a) \cup (b, \infty)$ is an open set, and so $[a,b]$ is closed.

In general, for some $([x_1, y_1] \times \cdots \times [x_n, y_n]) \subseteq \mathbb{R}^n$, we observe the complement is open, so closed boxes in $\mathbb{R}^n$ are closed (as per the name).

In the topological space endowed with the discrete topology, every subset is open, and so every subset is closed.

[2.6.3] Theorem

Let X be a topological space. Then:

(1) $\emptyset$ and X are closed.

(2) The finite union of closed sets is closed.

(3) The arbitrary intersection of closed sets is closed.

Proof.

(1) Note $\emptyset^C = X$ and $X^C = \emptyset$ are both open, so they're both closed.

(2) Let $A_1, \dots, A_n$ be finitely many closed sets. Then

$$\left(\bigcup_{k=1}^{n} A_k\right)^C = X \setminus \left(\bigcup_{k=1}^{n} A_k\right) = \bigcap_{k=1}^{n} (X \setminus A_k)$$

is the finite intersection of open sets, which is open. Thus the finite union is closed.

(3) Let $\{A_\lambda\}_{\lambda \in \Lambda}$ be a collection of closed sets. Then

$$\left(\bigcap_{\lambda \in \Lambda} A_k\right)^C = X \setminus \left(\bigcap_{\lambda \in \Lambda} A_k\right) = \bigcup_{\lambda \in \Lambda} (X \setminus A_k)$$

is the arbitrary union of open sets, which is open. Thus the arbitrary intersection is closed.

❦

Now we define a closed set with respect to a topology. If Y is a subspace of X, then a set $A \subseteq Y$ is closed in Y if A is closed in the subspace topology of Y (equivalently, if $Y \setminus A$ is open in Y).

[2.6.4] Theorem (*Criterion for Closed Set in Subspace*)

Let X be a topological space, and let $Y \subseteq X$ be a subspace. Then a subset $A \subseteq Y$ is closed if and only if A is the intersection of a closed subset of X with Y.

> **Proof.**
>
> $\Longrightarrow$: Suppose A is closed in Y. By definition, its complement $Y \setminus A$ is open in Y. By the definition of the subspace topology, there exists an open set U in X such that $Y \setminus A = U \cap Y$. Then,
> $$A = Y \setminus (Y \setminus A) = Y \setminus (U \cap Y) = Y \setminus U = Y \cap (X \setminus U).$$
> Since U is open in X, its complement $X \setminus U$ is closed in X. Letting $C = X \setminus U$, we have $A = C \cap Y$, where C is a closed set in X.
>
> $\Longleftarrow$: Suppose $A = C \cap Y$, where C is a closed set in X. We wish to show that A is closed in Y. Consider the complement of A in Y:
> $$Y \setminus A = Y \setminus (C \cap Y) = Y \setminus C = Y \cap \underbrace{(X \setminus C)}_{\text{open set in } X}.$$
> Since C is closed in X, its complement $X \setminus C$ is open in X. By the definition of the subspace topology, the intersection of an open set in X with Y is open in Y. Therefore, $Y \setminus A$ is open in Y, which implies that A is closed in Y, completing the backward direction. ❦

Note this is analogous to the fact that a set open in a subspace topology is simply an open set in the parent topology intersected with the subspace.

[2.6.5] Theorem

Let X be a topological space, and let $Y \subseteq X$ be a subspace. If a subset $A \subseteq Y$ is closed in Y, and Y is closed in X, then A is closed in X.

> **Proof.** A is the intersection of some closed subset of X with Y, which is also a closed subset of X—thus A is closed in X. ❦

[2.6.6] Definition (*Interior, Closure of Set*)

Let A be a subset of a topological space X. The **interior** of A is said to be the union of all open subsets of A, denoted $\mathrm{Int}(A)$. The **closure** of A is said the be the intersection of all closed subsets of X that contain A, denoted $\overline{A}$.

[2.6.7] Remark (*Relationship between Interior and Closure*)

Note the interior of a subset A of a topological space X is the union of every single open subset of A, which is strictly the largest open subset of A. Analogously, the closure of a subset A of a topological space X is the intersection of all closed subsets of X that contain A, which is strictly the smallest closed superset of A. Then
$$\mathrm{Int}(A) \subseteq A \subseteq \overline{A}.$$
Of course, if A is open, then $\mathrm{Int}(A) = A$, and analogously if A is closed, then $A = \overline{A}$.

Let A be a subset of the topological space X, but also a subset of a subspace $Y \subseteq X$. Of course, the closure of A as a subset of Y is not necessarily the same as the closure of A as a subset of X.

[2.6.8] Theorem (*Closure of Subset under Subspace*)

Suppose X is a topological space, Y is a subspace of X, and $A \subseteq Y \subseteq X$. Then the closure of A under Y, say $\mathrm{Cl}_Y(A)$ equals $\mathrm{Cl}_X(A) \cap Y$.

Proof. We show $\mathrm{Cl}_Y(A) = \mathrm{Cl}_X(A) \cap Y$ by proving a forward and backward inclusion.

$\subseteq$: Suppose $x \in \mathrm{Cl}_Y(A)$. Then note x is in every closed superset of A contained in Y, and of course $\mathrm{Cl}_X(A) \cap Y$ is one example, so $x \in \mathrm{Cl}_X(A) \cap Y$.

$\supseteq$: Suppose $x \in \mathrm{Cl}_X(A) \cap Y$. Then let C be any arbitrary closed superset of A such that $A \subseteq C \subseteq Y$. If $x \in C$, then it follows that $x \in \mathrm{Cl}_Y(A)$. Since C is closed in Y, and Y is a subspace of X (and thus endowed with the subspace topology), there is some closed subset $C_X \subseteq X$ such that $C = C_X \cap Y$. But $A \subseteq (C = C_X \cap Y)$, implying $A \subseteq C_X$. But $\mathrm{Cl}_X(A)$ is the minimal superset of A that is closed, so $A \subseteq \mathrm{Cl}_X(A) \subseteq C_X$. Thus $x \in C$, completing the proof. ❦

The definition of the closure of a set, while elegant, is impractical for explicit computation. For a topological space X and subset A, one would need to identify every closed subset of X containing A, then intersect them all—a collection far too large to work with directly. Instead, we may characterize membership in $\overline{A}$ in a more tractable ways: via the concept of *limit points*.

[2.6.9] Definition (*Neighborhood*)

A set U is said to be a **neighborhood** containing x if U is an open set containing x.

[2.6.10] Theorem (*Characterization of Closure of Set*)

Suppose A is a subset of the topological space X. Then the following statements are true.

(1) $x \in \overline{A}$ if and only if every neighborhood U containing x intersects A.

(2) $x \in \overline{A}$ if and only if each $B \in \mathcal{B}$—where $\mathcal{B}$ is a basis for the topology of X—containing x intersects A.

Proof.

(1) We proceed by contrapositive. That is, we show that $x \notin \overline{A}$ if and only if there exists some open neighborhood $U \subseteq X$ of x. First, suppose $x \notin \overline{A}$. Note $X \setminus \overline{A}$ is an open subset of X that must contain x, but does not intersect A by construction. Conversely, suppose some open neighborhood $U \subseteq X$ of x exists that doesn't intersect A. Then $X \setminus U$ is a closed superset of A that does not contain x, and so $x \notin \overline{A}$, completing the proof.

(2) Basis elements that contain x are open neighborhoods about x, and open neighborhoods about x that intersect A contain basis elements.

❦

[2.6.11] Remark (*Neighborhood Characterization of Closure*)

From real analysis, recall the open interval $A = (0,1) \subseteq \mathbb{R}$ (with $\mathbb{R}$ endowed with the standard topology), and note $\overline{A} = [0,1]$. Now consider $0 \in \overline{A}$. (1) from the previous theorem says that all neighborhoods about 0 must intersect A. Indeed, any open interval $(-\varepsilon,\varepsilon) \cap A \neq \emptyset$, and so 0 is in the closure for A.

For some set A, our theorem says that the closure $\overline{A}$ contains A, as well as additional points that, when taking an open neighborhood around, always intersect A. We formalize this idea and relate them back to the closure.

[2.6.12] Definition (*Limit Point*)

Suppose A is a subset of a topological space X. We say $x \in X$ is a **limit point** if every open neighborhood $U \subseteq X$ about x intersects A at some point *other than x itself*. That is, for every open set $U \subseteq X$ with $x \in U$, it follows that $U \cap (A \setminus \{x\}) \neq \emptyset$. Said differently, x is a limit point if it belongs to the closure of $A \setminus \{x\}$.

[2.6.13] Theorem (*Closure and Limit Points*)

Let A be a subset of the topological space X. Let A' be the set of all limit points of A. Then

$$\overline{A} = A \cup A'.$$

Proof. We prove equality by forward and backward inclusion.

$\subseteq$: Suppose $x \in \overline{A}$. If $x \in A$, then $x \in A \cup A'$. Otherwise, if $x \notin A$, then $A = A \setminus \{x\}$. But note that, since $x \in \overline{A}$, every open neighborhood $U \subseteq X$ of x intersects A nontrivially, so $U \cap (A \setminus \{x\}) \neq \emptyset$. Thus $x \in A' \subseteq A \cup A'$.

$\supseteq$: Suppose $x \in A \cup A'$. If $x \in A$, $x \in \overline{A}$, obviously. Now suppose $x \notin A'$, and so $x \in A'$. Analogous to before, this means $A \setminus \{x\} = A$. Since $x \in A'$, for every open neighborhood $U \subseteq X$ of x, $U \cap (A \setminus \{x\}) = U \cap A \neq 0$. Thus $x \in \overline{A}$, completing the proof. ❦

[2.6.14] Corollary

A subset of a topological space is closed if and only if it contains its limit points.

Proof. Let A be a subset of topological space X, and let A' be the set of limit points. First, suppose A is closed. Then $A = \overline{A} = A \cup A'$, meaning $A' \subseteq A$. Now suppose A contains all its limit points, and so $A' \subseteq A$. Then $A' \cup A \subseteq A$ and $A \subseteq A' \cup A$, meaning $A = A \cup A' = \overline{A}$, completing the proof. ❦

[2.6.15] Remark (*Prelude to Hausdorffness*)

Recall the standard topology on $\mathbb{R}$. A few nice properties are evident in its study.

(1) For each $x_0 \in \mathbb{R}$, the singleton $\{x_0\}$ is closed. For any $x : x \neq x_0$, we may construct an open neighborhood $U = (x-\varepsilon, x+\varepsilon)$ specifically to enforce $x_0 \notin U$ by choosing $\varepsilon > 0$ to be small enough.

(2) For any convergent real sequence $(x_n) \subseteq \mathbb{R}$, we say $x_n \to x$ if, for every open neighborhood U of x, there is some $N \in \mathbb{Z}_+$ such that $x_n \in U$ for every $n \geq N$. In $\mathbb{R}$, this limit is unique—that is, if $x_n \to x_1$ and $x_n \to x_2$, then $x_1 = x_2$.

These don't transfer to general topological spaces though. Consider the triplet $S = \{a,b,c\}$ endowed with topology $\{\varnothing, \{b\}, \{a,b\}, \{b,c\}, S\}$.

(1) We see that $\{b\}$ is not closed, since $S \setminus \{b\} = \{a,c\}$ is not in the topology (i.e., not open).

(2) Moreover, with $\mathbb{R}$'s definition of convergence, let's define $x_n = b$ and see what it converges to. For every open neighborhood of b ($\{a,b\}, \{b\}, \{b,c\}, \{a,b,c\}$), we see $x_n = b$ is in each neighborhood for every $n \in \mathbb{Z}_+$. The open neighborhoods of a ($\{a,b\}, \{a,b,c\}$) and c ($\{b,c\}, \{a,b,c\}$) also contain b. Thus the conclusion is that $\lim(x_n) \in \{a,b,c\}$, which does not agree with $\mathbb{R}$'s topological properties.

There is some property of $\mathbb{R}$'s standard topology that makes it nicer than S's topology, in that singletons are closed and convergent sequences have a unique limit.

[2.6.16] Definition (*Hausdorffness*)

A topological space X is said to be **Hausdorff** if, for any pair of distinct points $x_1, x_2 \in X$, there exists two disjoint neighborhoods $U_1, U_2 \subseteq X$ about x_1 and x_2 respectively.

[2.6.17] Theorem (*Singletons are Closed in Hausdorff Space*)

Every singleton subset of a topological space X is closed.

> **Proof.** Suppose $\{x_0\} \subseteq X$. Let $x \in X$ be such that $x \neq x_0$. By Hausdorffness of X, there are disjoint open neighborhoods $U_1, U_2 \subseteq X$ about x and x_0. Thus $x \notin \overline{A}$. This implies $\overline{A} = \{x_0\}$, meaning the singleton is closed. ❦

The property that singleton subsets of a topological space are closed is weaker than the space being Hausdorff. We extract this condition as the T_1 **axiom**.

[2.6.18] Definition (*T_1 Space*)

A topological space X is said to be T_1 if every singleton subset of X is closed.

From a geometric perspective, a space is T_1 if for any two distinct points x, y, there exists an open neighborhood of x that does not contain y. The space is Hausdorff if, for any two distinct points x and y, there exist disjoint open neighborhoods of x and y. Thus Hausdorffness is a stronger condition than T_1. We will see shortly that although T_1 spaces preserve our intuition about limit points, T_1 alone is not strong enough to guarantee the uniqueness of sequence limits, whereas Hausdorff spaces do.

[2.6.19] Theorem

Let A be a subset of the topological space X satisfying the T_1 axiom. Then x is a limit point of A if and only if every neighborhood of X contains infinitely many points of A.

Proof.

($\Longrightarrow$): Suppose x is a limit point of A. Then there is some open neighborhood $U \subseteq X$ of x such that $(U \cap (A \setminus \{x\}) = S) \neq \emptyset$. Suppose S is finite, so $S = \{x_1, \ldots, x_n\}$. Note that S is a finite union of singleton subsets of X, and since singletons are closed in T_1 spaces, S is closed. Thus $X \setminus S$ is open. Let $U' = U \cap (X \setminus S)$. Then U' is open, $x \in U$, but $U \cap (A \setminus \{x\}) = \emptyset$. We have constructed an open neighborhood about x that does not intersect $A \setminus \{x\}$, meaning x is not a limit point, forcing a contradiction. Thus $U \cap (A \setminus \{x\})$ is infinite.

($\Longleftarrow$): Every open neighborhood about x intersects A infinitely amount of times, so every open neighborhood about x

intersects $A \setminus \{x\}$ infinitely many times, meaning x is a limit point. ❦

[2.6.20] Theorem (*Uniqueness of Limits in Hausdorff Space*)

Suppose X is a Hausdorff topological space. Then a sequence $(x_n)_{n=1}^{\infty} \subseteq X$ converges to, at most, one limit point.

Proof. Suppose x_n converges, so $x_n \to x$. Let $x_0 \neq x$: we show $x_n \not\to x_0$, confirming x is the only limit for the sequence. Since $x \neq x_0 \in X$, a Hausdorff space, we can construct disjoint open neighborhoods U and U_0 about x and x_0 respectively such that $U \cap U_0 = \emptyset$. Recall that since $x_n \to x$, there exists some $N \in \mathbb{Z}_+$ such that $x_n \in U$ for every $n \geq N$. But since U_0 is disjoint from U, it follows that the only terms of the sequence U_0 can possibly contain are $x_1, \ldots, x_{N-1}$. In other words, the tail for (x_n) is not contained in U_0, so $x_n \not\to x_0$, completing the proof. ❦

2.6.1 Problems

[2.6.21] Problem

Let X be an ordered set endowed with the order topology. Show that $\overline{(a,b)} \subseteq [a,b]$. Under what conditions does equality hold?

Proof. Note $\overline{(a,b)}$, by definition, is the smallest closed superset of (a,b). Thus, if $[a,b]$ is a closed superset of (a,b), then $\overline{(a,b)} \subseteq [a,b]$ follows immediately. Note that $[a,b]^c = (\leftarrow, a) \cup (b, \rightarrow)$ is a union of two open sets (so it's open), making $[a,b]$ closed, completing the proof.

Equality is achieved when $[a,b] \subseteq \overline{(a,b)}$, since we have forward and backward inclusion. Since $[a,b] = (a,b) \cup \{a\} \cup \{b\}$, simply knowing that a, b are limit points of (a,b) forces equality. We show a (resp. b) is a limit point of (a,b) if a (resp. b) has no immediate successor (resp. immediate predecessor).

Suppose a has no immediate successor. Let $U \subseteq X$ be an open neighborhood about a. Then note that we can locally refine U by choosing some basis element of the order topology, V, such that $a \in V \subseteq U$. Note a basis element is of the form $V = (a,c)$, where $c : a < c \leq b$. In any case, since a has no immediate successor, there is some $x < c$ for which $x \in V \subseteq U$. This means $x \in U \cap ((a,b) \setminus \{a\}) = U \cap (a,b)$, meaning a is a limit point for (a,b).

[2.6.22] Problem

Let A, B be subsets of a topological space X. Prove the following.

(1) If $A \subseteq B$, then $\overline{A} \subseteq \overline{B}$.

Proof.

(1) Suppose $A \subseteq B$. Let $x \in \overline{A}$. If $x \in A$, then $x \in B \subseteq \overline{B}$ trivially, so suppose $x \notin A$. Then x is a limit point, so every open neighborhood $U \subseteq X$ about x is such that $U \cap (A \setminus \{x\}) \neq \emptyset$. But $A \subseteq B$, so this implies $U \cap (B \setminus \{x\}) \neq \emptyset$, meaning x is a limit point of B too, so $x \in \overline{B}$.

[2.6.23] Problem

Show that every order topology is Hausdorff.

Proof. Suppose an ordered set X is endowed with the order topology. Let $x \neq y \in X$, and without loss of generality, suppose $x < y$. Suppose y is x's immediate successor: then there exist open neighborhoods $U = (-\infty, y) \ni x$ and $V = (x, \infty) \ni y$ where $U \cap V = \emptyset$. If y is not x's immediate successor, there exists $t : x < t < y$. Then $U = (-\infty, t) \ni x$ and $V = (t, \infty) \ni y$ such that $U \cap V = \emptyset$, completing the proof.

[2.6.24] Problem

Show the product of two Hausdorff spaces is Hausdorff.

Proof. Let $(x_1, y_1) \neq (x_2, y_2) \in X \times Y$, where X and Y are Hausdorff topological spaces. Without loss of generality, suppose $x_1 \neq x_2$. By Hausdorffness of X, there exists open neighborhoods $U_X \ni x_1$ and $V_X \ni x_2$ such that $U_X \cap V_X = \emptyset$. There also exists some open neighborhood $V \ni y_1, y_2$. Then $(U_X \times V) \ni (x_1, y_1)$ and $(V_X \times V) \ni (x_2, y_2)$ are disjoint open neighborhoods of $X \times Y$, completing the proof.

[2.6.25] Problem

Show the subspace of a Hausdorff space is Hausdorff.

Proof. Suppose X is a Hausdorff space and $Y \subseteq X$ is a subspace. Let $x \neq y \in Y$. Note there exists open neighborhoods $U \subseteq X$ [illegible] and $V \cap Y$. Of course, $x \in U \cap Y$ and $y \in V \cap Y$, and note $(U \cap Y) \cap (V \cap Y) = (U \cap V) \cap Y = \emptyset \cap Y = \emptyset$, completing the proof. ❦

[2.6.26] Problem

Show that a topological space X is Hausdorff if and only if the diagonal $\Delta = \{(x, x) : x \in X\}$ is closed in $X \times X$.

Proof.

($\Longrightarrow$): Suppose X is Hausdorff. Suppose $(x, y) \in X \times X$ is a limit point for Δ. Then, for every open neighborhood $U \subseteq X \times X$ of (x, y), it follows that $U \cap (\Delta \setminus (x, y)) \neq \emptyset$. For the sake of contradiction, suppose $x \neq y$. Then, by Hausdorffness of X, there exist disjoint open neighborhoods $U \subseteq X$ and $V \subseteq X$ such that $x \in U$ and $y \in V$. Note then that $U \times V$ is open in $X \times X$. Since U and V are disjoint, there are no tuples with the same entry in each slot when taking their Cartesian product—that is, $(U \times V) \cap (\Delta \setminus (x, y)) = \emptyset$, which is a contradiction. Thus $x = y$, implying Δ contains all its limit points and thus is closed.

($\Longleftarrow$): Suppose Δ is closed, and so Δ^c is open. Let $x \neq y \in X$, then observe $(x, y) \in \Delta^c$. Since Δ^c is an open subset of $X \times X$, there is some basis element $U \times V$ (where U, V are open subsets of X) such that $(x, y) \in U \times V \subseteq \Delta^c$. For $U \times V$ to be a subset of Δ^c, this implies no elements of $U \times V$ may be in Δ: thus $U \cap V = \emptyset$. Thus we have constructed two disjoint open neighborhoods that contain x and y respectively, completing the proof. ❦

[2.6.27] Problem

Endow $\mathbb{R}$ with the finite complement topology. For what points of $\mathbb{R}$ does $x_n = 1/n$ converge to.

Proof. Note a set $O \subseteq \mathbb{R}$ is open if $O = \emptyset$, or $\mathbb{R} \setminus O$ is finite (i.e., it misses finitely many points of $\mathbb{R}$). Let $x \in \mathbb{R}$ be arbitrary, and let $O \subseteq \mathbb{R}$ be an arbitrary open neighborhood such that $x \in O$. Note O can only miss finitely many elements of $\mathbb{R}$, so O must contain infinitely many elements of x_n. That is, there is some $N \in \mathbb{Z}_+$ such that $x_n \in O$ for every $n \geq N$. Thus $x_n \to x$, but x is arbitrary, so (x_n) converges to every real number. ❦

[2.6.28] Definition (*Boundary*)

Suppose X is a topological space and $A \subseteq X$. We define the boundary of A to be the set

$$\partial A = \overline{A} \cap \overline{X \setminus A}.$$

[2.6.29] Remark (*Intuition Behind Boundary*)

Suppose X is a topological space and $A \subseteq X$. Recall that if $x \in \overline{A}$, then every open neighborhood $U \subseteq X$ about x is such that $U \cap A \neq \emptyset$. Equivalently, if $x \in \overline{X \setminus A}$, then every open neighborhood $U \subseteq X$ about x is such that $U \cap (X \setminus A) \neq \emptyset$. Thus, if $x \in (\overline{A} \cap \overline{X \setminus A}) = \partial A$, every open neighborhood $U \subseteq X$ about x must have nontrivial intersection with A and $X \setminus A$. Graphically, drawing an open neighborhood about a point leads to it spilling inside and outside the set.

[2.6.30] Problem

Suppose X is a topological space and $A \subseteq X$. Show that $\mathrm{Int}(A)$ and ∂A are disjoint.

Proof. Suppose $x \in \mathrm{Int}(A)$. Since $\mathrm{Int}(A) \subseteq A$, it follows that $x \in A$. For the sake of contradiction, suppose $x \in \partial A$, implying $x \in \overline{X \setminus A}$. Note $\overline{X \setminus A} = (X \setminus A) \cap (X \setminus A)'$ (where S' refers to the limit points of S). Since $x \in A$, $x \notin (X \setminus A)$, so it is only possible that $x \in (X \setminus A)'$. This implies that every open subset $U \subseteq X$ must be such that $U \cap (A^c \setminus \{x\}) \neq \emptyset$. But note $x \in \mathrm{Int}(A) \subseteq A$, meaning every open neighborhood of x must be fully contained in A, which is a contradiction. Thus $\mathrm{Int}(A) \cap \partial A = \emptyset$. ❦

[2.6.31] Problem

Suppose X is a topological space and $A \subseteq X$. Show $\overline{A} = \mathrm{Int}(A) \cup \partial A$. That is, $\overline{A}$ is the disjoint union of the interior and boundary of A.

Proof. We prove this by forward and backward inclusion. First, suppose $x \in \overline{A}$, and suppose $x \notin \mathrm{Int}(A)$. This implies there are no X-open neighborhoods of x that are subsets of A. Indeed, this implies the only X-open neighborhoods of x must intersect $X \setminus A$. Thus $x \in \overline{X \setminus A}$, so $x \in \partial A$. Thus every $x \in \overline{A}$ must be either in $\mathrm{Int}(A)$ or ∂A, and so $\overline{A} \subseteq \mathrm{Int}(A) \cup \partial A$. Conversely, suppose $x \in \mathrm{Int}(A) \cup \partial A$. If $x \in \mathrm{Int}(A)$, then $x \in A \subseteq \overline{A}$. If $x \in (\partial A = \overline{A} \cap \overline{X \setminus A})$, then $x \in \overline{A}$, completing the proof. ❦

[2.6.32] Problem

Suppose X is a topological space and $A \subseteq X$. Show that $\partial A = \emptyset$ if and only if A is clopen.

Proof. Suppose A is X-clopen. Since A is closed, $A = \overline{A}$. Let $x \in A = \overline{A}$ be a limit point of A. Then note every X-open neighborhood $O \ni x$ is such that $O \subseteq A$, so no open neighborhoods intersect $X \setminus A$. Thus $x \notin \overline{X \setminus A}$, implying $\partial A = \emptyset$. Conversely, suppose $\partial A = \overline{A} \cap \overline{X \setminus A} = \emptyset$. If x is a limit point of A, then $x \in \overline{A}$, then it cannot be in $X \setminus A$ and so $x \in A$, meaning A is closed. The same argument can be made to show $X \setminus A$ is also closed, meaning A is open as well, completing the proof. ❦

[2.6.33] Problem

Suppose X is a topological space and $A \subseteq X$. Show that U is open if and only if $\partial U = \overline{U} \setminus U$.

Proof. ($\Longrightarrow$): Suppose U is open. We prove the forward direction by forward and backward inclusion. To get started, suppose $x \in \overline{U} \setminus U$. Then note $x \notin U$, and so $x \in X \setminus U \subseteq \overline{X \setminus U}$, and so $x \in \partial U$. Conversely, suppose $x \in \partial U$. Immediately, we know $x \in \overline{U}$. For the sake of contradiction, suppose $x \in U$. Since U is open, there is some basic X-open neighborhood O such that $x \in O \subseteq U$. But $O \cap (X \setminus U) = \emptyset$, so $x \notin \overline{X \setminus U}$, a contradiction. Thus $x \notin U$, meaning $x \in \overline{U} \setminus U$. Thus $\partial U = \overline{U} \cap \overline{X \setminus U} = \overline{U} \setminus U$.

($\Longleftarrow$): Suppose $\partial U = \overline{U} \setminus U$. To show U is open, we show $X \setminus U$ is closed, or that $X \setminus U = \overline{X \setminus U}$. Let $x \in \overline{X \setminus U}$ be a limit point for $X \setminus U$. Note that if $x \in U$, then $x \in \overline{U}$ and so $x \in \overline{U} \cap \overline{X \setminus U}$, but $x \notin U \setminus \overline{U}$, which is contradictory—thus $x \in X \setminus U$, completing the proof. ❦

2.7 Continuous Functions

[2.7.1] Definition (*Continuous Function*)

A set-function $f : X \to Y$ is said to be **continuous** if, for every open subset $V \subseteq Y$, the preimage $f^{-1}(V)$ is an open subset of X.

Note continuity depends on the topologies endowed on the domain and codomain of the function.

[2.7.2] Remark (*Continuity of Basis and Subbasis Elements*)

Suppose the topology on Y has a basis $\mathcal{B}$. Then any open subset $V \subseteq Y$ can be written as

$$V = \bigcup_{\lambda \in \Lambda} B_\lambda,$$

where $\{B_\lambda\}_{\lambda \in \Lambda}$ is an arbitrary collection of basis elements. Then note

$$f^{-1}(V) = f^{-1}\left(\bigcup_{\lambda \in \Lambda} B_\lambda\right) = \bigcup_{\lambda \in \Lambda} f^{-1}(B_\lambda),$$

so if the preimage of each basis element is open, then the function is continuous on its entire domain. Similarly, note that $\mathcal{B}$ can have a subbasis $\mathcal{S}$, so any B_λ can be written as

$$B_\lambda = \bigcap_{k=1}^{n} S_k,$$

where $\{S_1, \ldots, S_n\} \subseteq \mathcal{S}$ is a collection of subbasis elements. Then

$$f^{-1}(B_\lambda) = f^{-1}\left(\bigcap_{k=1}^{n} S_k\right) = \bigcap_{k=1}^{n} f^{-1}(S_k),$$

so if the preimage of each subbasis element is open, then the function is continuous on its entire domain.

[2.7.3] Example (*Continuity of Single-Variable Real Functions*)

Consider some function $f : \mathbb{R} \to \mathbb{R}$, where the domain and codomain are endowed with the standard topology.

First, suppose f is continuous with our definition utilizing the preimage of open sets. Fix some $x_0 \in \mathbb{R}$ and let $\varepsilon > 0$ be arbitrary. Then note $V = (f(x_0) - \varepsilon, f(x_0) + \varepsilon)$ is a basis element for $\mathbb{R}$, so it is open in the codomain. Of course, $x_0 \in f^{-1}(V)$. Since $f^{-1}(V)$ is an open subset of $\mathbb{R}$, we can locally refine by some basis element (a, b) such that $x_0 \in (a, b) \subseteq f^{-1}(V)$. Thus there is some $\delta > 0$ such that $(x_0 - \delta, x_0 + \delta) \subseteq f^{-1}(V)$. Indeed, this implies the $\varepsilon - \delta$ variant of continuity at x_0: for any $\varepsilon > 0$, there is some $\delta > 0$ such that for every $x \in (x_0 - \delta, x_0 + \delta)$, it is true that $f(x) \in (f(x_0) - \varepsilon, f(x_0) + \varepsilon)$.

Now suppose f is continuous with the $\varepsilon - \delta$ definition. That is, for any $x_0 \in \mathbb{R}$ and $\varepsilon > 0$, there is some $\delta > 0$ such that

$$|x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon.$$

This can be reformulated into open and closed sets quite easily:

$$x \in B_\delta(x_0) \implies f(x) \in B_\varepsilon(f(x_0)).$$

Thus f is continuous if every open set containing $f(x_0)$ has a preimage that is an open set. Thus these two characterizations are equal.

[2.7.4] Theorem (*Characterizations of Continuity*)

Let X, Y be topological spaces, and let $f : X \to Y$. Then the following are equivalent.

(1) f is continuous.

(2) For every subset $A \subseteq X$, it follows that $f(\overline{A}) \subseteq \overline{f(A)}$.

(3) For every closed subset $B \subseteq Y$, the preimage $f^{-1}(B) \subseteq X$ is closed as well.

(4) For every $x \in X$ and for every open neighborhood $V \subseteq Y$ of $f(x)$, there is some open neighborhood $U \subseteq X$ of x where $f(U) \subseteq V$.

Proof. We prove (1) $\implies$ (2) $\implies$ (3) $\implies$ (1) and (1) $\implies$ (4) $\implies$ (1).

(1) $\implies$ (2): Suppose f is continuous. We seek to show that, for any $x \in \overline{A}$, it is true that $f(x) \in \overline{f(A)}$. To get started, consider an open neighborhood $V \subseteq Y$ about $f(x) \in f(\overline{A})$. Then note $f^{-1}(V) \subseteq X$ is open by continuity. Since $x \in \overline{A}$, all open neighborhoods of x intersect A. Since $x \in f^{-1}(V)$ and $f^{-1}(V)$ is open, it follows that there is some $y \in f^{-1}(V) \cap A$. Then note $f(y) \in V \cap f(A)$. Thus arbitrary open neighborhoods of $f(x)$ intersect $f(A)$, implying $f(x)$ is in the closure of $f(A)$. Thus $f(x) \in \overline{f(A)}$, and ultimately $f(\overline{A}) \subseteq \overline{f(A)}$.

(2) $\implies$ (3): Suppose that, for every $A \subseteq X$, it is true that $f(\overline{A}) \subseteq \overline{f(A)}$. Let $B \subseteq Y$ be closed, and let $A = f^{-1}(B)$—we wish to show A is closed, and we do this by showing $A = \overline{A}$. Trivially, we note $A \subseteq \overline{A}$, so we show $\overline{A} \subseteq A$. As such, let $x \in \overline{A}$. Then note

$$f(x) \in f(\overline{A}) \subseteq \overline{f(A)} = \overline{f(f^{-1}(B))} \subseteq \overline{B} = B.$$

Thus $x \in f^{-1}(B) = A$, meaning $A = \overline{A}$, meaning $f^{-1}(B)$ is closed.

(3) $\implies$ (1): Suppose that, for every closed set $B \subseteq Y$, the preimage $f^{-1}(B) \subseteq X$ is closed as well. Let $O \subseteq Y$ be open. Then note $Y \setminus O$ is closed in Y, and so $f^{-1}(Y \setminus O)$ is closed in X. Thus $X \setminus f^{-1}(Y \setminus O) = f^{-1}(O)$ is open, meaning f is continuous.

(1) $\implies$ (4): Suppose f is continuous. Fix $x \in X$, then let $V \subseteq Y$ be an open neighborhood of $f(x)$. Since f is continuous, $U = f^{-1}(V) \subseteq X$ is an open neighborhood of x. Then note $f(U) = f(f^{-1}(V)) \subseteq V$ as desired.

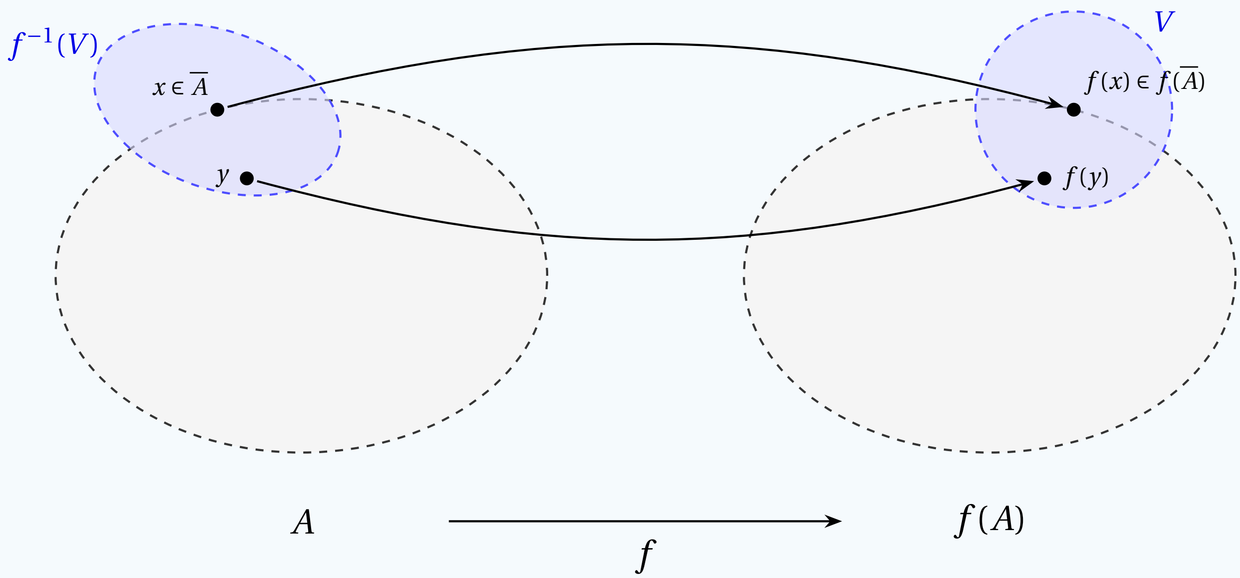
(4) $\implies$ (1): Let $V \subseteq Y$ be an arbitrary open set. We seek to show $f^{-1}(V) \subseteq X$ is open. If $f^{-1}(V) = \varnothing$, then we are done. Otherwise, let $x \in f^{-1}(V)$. Then $f(x) \in V$, but there is some open neighborhood $U_x \subseteq X$ of x such that $f(U_x) \subseteq V$. Note then that $U_x \subseteq f^{-1}(V)$ for each $x \in f^{-1}(V)$, and so $\bigcup_{x \in f^{-1}(V)} U_x \subseteq f^{-1}(V)$. Moreover, $\{U_x\}_{x \in f^{-1}(V)}$ covers $f^{-1}(V)$, so $f^{-1}(V) \subseteq \bigcup_{x \in f^{-1}(V)} U_x$. Thus,

$$f^{-1}(V) = \bigcup_{x \in f^{-1}(V)} U_x,$$

which is the arbitrary union of open sets, completing the proof. ❦

[2.7.5] Remark (*Intuition Behind Characterizations of Continuity*)

The statement in (2), that $f(\overline{A}) \subseteq \overline{f(A)}$ for every $A \subseteq X$, can be interpreted locally and globally. In the case of metric spaces, if there is some $x \in \overline{A}$, then there is some sequence $(x_n) \subseteq A$ such that $x_n \to x$. If f is continuous, then the image sequence $f(x_n) \subseteq f(A)$ is such that $f(x_n) \to f(x)$. Thus elements of A and limit points of A are sent to elements of $f(A)$ and limit points of $f(A)$. That is, the image of the closure of A lives in the closure of $f(A)$, encoding the idea that $f(\overline{A}) \subseteq \overline{f(A)}$. This intuition locally reasons why sequential continuity works, and globally reasons that the continuous image of the closure of A lives in the closure of the continuous image of A. The global principle is illustrated in this schematic.



Statement (3) is a statement that is dual to the definition. Since open sets are dual to closed sets under complementation, it makes sense that changing open sets to closed ones in the definition of continuity yields the same characterization, since open subsets of a topological space are in bijective correspondence with closed subsets.

Statement (4) is a local characterization of continuity. In fact, we say a function f is continuous specifically at some point x_0 if, for every open neighborhood $V \subseteq Y$ of $f(x_0)$, there is some open neighborhood $U \subseteq X$ of x_0 such that $f(U) \subseteq V$. If this statement is true for all elements in the domain of f, then f is continuous on its domain. This statement is reminiscent of the challenge-response system used in $\varepsilon - \delta$ proofs in analysis. For a fixed element of the domain x and some neighborhood in the codomain about $f(x)$ (the “challenge”), we seek a neighborhood in the domain about x whose image lies in the neighborhood in the codomain (the “response”).

[2.7.6] Definition (*Homeomorphism, Homeomorphic Spaces*)

Suppose X, Y are topological spaces and $f : X \to Y$ is a set-function. If f is bijective, and $f : X \to Y$ is continuous, and $f^{-1} : Y \to X$ is continuous, we say f is a **homeomorphism**. Two spaces X, Y are **homeomorphic** if a homeomorphism exists between them.

[2.7.7] Remark (*Equivalent Characterization of Homeomorphism*)

Recall $f : X \to Y$ is continuous if, for every open set $U \subseteq Y$, $f^{-1}(U) \subseteq X$ is open. Analogously, $f^{-1} : Y \to X$ is open if, for every open set $U \subseteq X$, $f(U) \subseteq Y$ is open. Thus a bijection $f : X \to Y$ such that $f(U)$ is open if and only if U is open is an alternate way to define a homeomorphism.

Note this means open sets between two homeomorphic spaces are in bijective correspondence. This property means homeomorphic spaces are not only identical as sets, but also identical as topological spaces! The structure of the two spaces are the same, and operations involving the topology of each space are analogous in each space. Homeomorphisms are akin to isomorphisms in that homeomorphic/isomorphic spaces are identical in structure and shape, up to labeling of elements.

[2.7.8] Definition (*Topological Embedding*)

Suppose $f : X \to Y$ is an injective, continuous map. Then let $\tilde{f} : X \to f(X)$ be the induced bijection from X to the image set $f(X)$. If $\tilde{f}$ is a homeomorphism, we say f is a **topological embedding** of X into Y. In other words, f is a topological embedding if it's an homeomorphism from its domain to its image.

We now list a few common types of continuous functions between topological spaces.

[2.7.9] Theorem (*Common Continuous Functions*)

Let X, Y, Z be topological spaces. Then the following functions are continuous.

(1) **Constant Function.** If $f : X \to Y$ maps every element of X to some $y_0 \in Y$, then f is continuous.

(2) **Inclusion Map.** If A is a subspace of X, the inclusion map $\iota : A \to X$ is continuous.

(3) **Composition.** If $f : X \to Y$ and $g : Y \to Z$ are both continuous functions, then $g \circ f : X \to Z$ is continuous.

(4) **Domain Restriction.** If $f : X \to Y$ is continuous and A is a subspace of X, then the restriction $f|_A : A \to Y$ is continuous.

(5) **Codomain Restriction/Expnasion.** If $f : X \to Y$ is continuous and Z is a subspace of Y that contains $f(X)$, then $g : X \to Z$ is continuous. Likewise, if Z is a superspace of Y, then $h : X \to Z$ is continuous as well.

(6) **Local Continuity.** The map $f : X \to Y$ is continuous if, for every $x \in X$, there is some neighborhood $U_x \subseteq X$ such that $f|_{U_x} : U_x \to Y$ is continuous.

[2.7.10] Theorem (*Pasting Lemma*)

Suppose $X = A \cup B$, where A, B are closed subsets of X. Let $f : A \to Y$ and $g : B \to Y$ be continuous functions such that $f(x) = g(x)$ for every $x \in A \cap B$. Define the function

$$h : X \to Y \quad h(x) = \begin{cases} f(x) & x \in A, \\ g(x) & x \in B. \end{cases}$$

Then h is continuous.

Proof. Let $O \subseteq Y$ be a closed subset of Y. Then note

$$h^{-1}(O) = f^{-1}(O) \cup g^{-1}(O).$$

By continuity, note $f^{-1}(O)$ is closed in A. By definition, this means $f^{-1}(O) = C \cap A$, where C is some closed subset of X. But since A is also a closed subset of X, this makes $f^{-1}(O)$ a closed subset of X. An entirely symmetric argument reveals $g^{-1}(O)$ is closed as well. Thus $h^{-1}(O)$ is closed in X, completing the proof. ❦

[2.7.11] Theorem (*Maps into Cartesian Product*)

Let $f : A \to X \times Y$ be a function defined by $f(x) = (f_X(x), f_Y(x))$. Then f is continuous if and only if $f_X : A \to X$ and $f_Y : A \to Y$ are both continuous.

Proof.

($\Longrightarrow$): Suppose f is continuous. Let $U \subseteq X$ be open: we seek to show $f_X^{-1}(U)$ is open as well. Note that $f_X = \pi_X \circ f$, so then $f_X^{-1} = (\pi_X \circ f)^{-1} = f^{-1} \circ (\pi_X)^{-1}$. Note that $\pi_X^{-1}(U) = U \times Y$, which is an open subset of $X \times Y$ when gifted with the product topology. Then, by continuity of f, the preimage of $U \times Y$ is open, meaning f_X^{-1} maps open sets to open sets, making f_X continuous. An entirely symmetric argument proves f_Y is continuous as desired.

($\Longleftarrow$): Suppose f_X, f_Y are continuous. Let $U \subseteq X \times Y$ be an open: we seek to show $f^{-1}(U)$ is open. Let $U = \bigcup_{\lambda \in \Lambda}(X_\lambda \times Y_\lambda)$, where $\{X_\lambda\}_{\lambda \in \Lambda}$ is a family of open subsets of X and $\{Y_\lambda\}_{\lambda \in \Lambda}$ is a family of open subsets in Y. Now note

$$\begin{aligned}
f^{-1}(U) &= \{x \in A : f(x) \in U\} \\
&= \left\{ x \in A : f(x) \in \bigcup_{\lambda \in \Lambda} (X_\lambda \times Y_\lambda) \right\} \\
&= \bigcup_{\lambda \in \Lambda} \{x \in A : f_X(x) \in X_\lambda, f_Y(x) \in Y_\lambda\} \\
&= \bigcup_{\lambda \in \Lambda} \Big(\{x \in A : f_X(x) \in X_\lambda\} \cap \{x \in A : f_Y(x) \in Y_\lambda\} \Big) \\
&= \bigcup_{\lambda \in \Lambda} \Big(f_X^{-1}(X_\lambda) \cap f_Y^{-1}(Y_\lambda) \Big).
\end{aligned}$$

By continuity of f_X and f_Y, we note each $f_X^{-1}(X_\lambda) \cap f_Y^{-1}(Y_\lambda)$ is the intersection of two open sets, and is therefore open. Because $f^{-1}(U)$ is a union of such open sets, it is also open, meaning f is continuous. ❦

2.7.1 Exercises

[2.7.12] Problem

Suppose $f: X \to Y$ is continuous. Suppose x is a limit point of the subset $A \subseteq X$. Is it true that $f(x)$ is a limit point of $f(A)$?

Proof. Suppose x is a limit point of A. Let $V \subseteq Y$ be some open neighborhood of $f(x)$. By continuity of f, note there is some open neighborhood $U \subseteq X$ of x such that $f(U) \subseteq V$. Then note $U \cap (A \setminus \{x\}) \neq \emptyset$ by the definition of a limit point. If our scenario is such that $f(U \cap (A \setminus \{x\})) = \{f(x)\}$ for *every single* open neighborhood of x, then $f(x)$ is not a limit point of $f(A)$.

Let $U = X$. Then $f(U \cap (A \setminus \{x\})) = f(A \setminus \{x\}) = \{f(x)\}$ would make what we seek false. A candidate for disproof is a constant function. Let $f: \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = 0$, then let $A = [0,1] \subseteq \mathbb{R}$. Of course, 0 is a limit point for A, but $f(A) = \{0\}$, and singletons don't have limit points, completing the disproof. ❦

[2.7.13] Problem

Let $y_0 \in Y$. Define $f: X \to X \times Y$ by $f(x) = (x, y_0)$. Show that f is a topological embedding.

Proof. Note $f(X) = X \times \{y_0\}$, so we seek to show $\tilde{f}: X \to X \times \{y_0\}$ is a homeomorphism, where the codomain is blessed with the subspace topology. $\tilde{f}$ is obviously bijective, so we simply show $\tilde{f}$ and $\tilde{f}^{-1}$ are continuous.

First, suppose $V \subseteq X \times \{y_0\}$ is open. Then $V = V' \cap (X \times \{y_0\})$, where $V' \subseteq X \times Y$ is open. Note that

$$V' = \bigcup_{\lambda \in \Lambda} (X_\lambda \times Y_\lambda),$$

where $\{X_\lambda\}$ and $\{Y_\lambda\}$ are families of open subsets of X and Y respectively. Then

$$V = \left[\bigcup_{\lambda \in \Lambda} (X_\lambda \times Y_\lambda)\right] \cap \left[X \times \{y_0\}\right] = \underbrace{\left(\bigcup_{\lambda \in \Lambda, y_0 \in Y_\lambda} X_\lambda\right)}_{:= O_X} \times \{y_0\},$$

where $O_X \subseteq X$ is open. Then

$$\tilde{f}^{-1}(V) = \{x \in X : \tilde{f}(x) \in O_X \times \{y_0\}\} = \{x : x \in O_X\} = O_X,$$

an open subset of X.

Now suppose $U \subseteq X$ is open. Then

$$\tilde{f}(U) = U \times \{y_0\} = \underbrace{(U \times Y)}_{\text{open subset of } X \times Y} \cap (X \times \{y_0\}),$$

which is an open set in $X \times \{y_0\}$. Thus $\tilde{f}$ is a homeomorphism, making f an embedding. ❦

[2.7.14] Problem

Let Y be an ordered set endowed with the order topology. Let $f, g: X \to Y$ be continuous. Show the set $A = \{x : f(x) \leq g(x)\}$ is closed in X. Then, show $h(x) = \min\{f(x), g(x)\}$ is continuous.

Proof. We show $A = \overline{A}$. Let $x \in \overline{A}$ be a limit point of A. For the sake of contradiction, suppose $x \notin A$, and so $f(x) > g(x)$. If there is some $c \in Y$ such that $f(x) > c > g(x)$, then let $V_g = (-\infty, c)$ and $V_f = (c, \infty)$. Otherwise, let $V_f = (g(x), \infty)$ and $V_g = (-\infty, f(x))$. Then $V_f, V_g \subseteq Y$ are disjoint, open neighborhoods of $f(x)$ and $g(x)$, where all elements of V_f are strictly greater than all elements of V_g. By continuity, there exist open neighborhoods $U_f, U_g \subseteq X$ of x such that $f(U_f) \subseteq V_f$ and $g(U_g) \subseteq V_g$. But note $U_f \cap U_g$ is an open neighborhood of x. Since x is a limit point, this implies there is some $u \in (U_f \cap U_g) \cap (A \setminus \{x\})$. Since $u \in A$, it follows that $f(u) \le g(u)$. But also note $f(u) \in V_f$ and $g(u) \in V_g$, meaning $f(u) > g(u)$ by how V_f, V_g are defined. BY THE ***TRICHOTOMY POSTULATE OF ORDERED SETS***, this is a contradiction. Thus $f(x) \le g(x)$, and so $x \in A$, making A closed. An entirely symmetric argument shows $B = \{x : f(x) \ge g(x)\}$ is closed in X.

Define $\tilde{f} : A \to Y$ and $\tilde{g} : B \to Y$. By the pasting lemma, we may paste together these maps to form $h : A \cup B \to Y$ such that

$$h(x) = \begin{cases} f(x) & x \in A \\ g(x) & x \in B \end{cases} = \min\{f(x), g(x)\},$$

which is continuous as desired.

[2.7.15] Problem

Let $\{A_\alpha\}$ be a collection of subsets of X such that $X = \bigcup_\alpha A_\alpha$. Let $f : X \to Y$ be such that $f|_{A_\alpha}$ for each α.

(1) Supposing $\{A_\alpha\}$ is a finite family of closed sets, show f is continuous.

(2) Construct an example where $\{A_\alpha\}$ is a countable family of closed sets and f is discontinuous.

Proof.

- For our base case, suppose there are two sets in the family, A_1 and A_2. Note $f|_{A_1}$ and $f|_{A_2}$ are continuous and agree on mutual points in their domain, so gluing them together yields a continuous function, which is f. Now suppose the hypothesis is true for $n-1$ many sets. Let $A' = A_1 \cup A_2 \cup \cdots \cup A_{n-1}$, which is closed (since finite unions of closed sets are closed). $f|_{A'}$ is continuous by inductive hypothesis. Then note $f|_{A_n}$ is also continuous by the original hypothesis, and $f|_{A'}, f|_{A_n}$ agree on common points of their domain. Thus f is continuous by the Pasting Lemma.
- Let $X = [0,1]$. Let $\{A_n\}_{n \in \mathbb{Z}_{\ge 0}}$ be defined such that $A_0 = \{1\}$, $A_1 = \{0\}$, and $A_n = A_{n-1} \cup [0, 1 - 1/n]$ for every $n \ge 2 \in \mathbb{Z}_+$. Then each A_n is closed, since the finite union of closed intervals is closed, and $[0,1] = \bigcup_{n \in \mathbb{Z}_{\ge 0}} A_n$. Define $f : [0,1] \to \mathbb{R}$ by $f(x) = 0$ for all $x \in [0,1)$, and $f(1) = 1$ otherwise. Then note $f|_{A_n}$ is continuous for each $n \in \mathbb{Z}_+$, since f is constant on this restriction. But f is not continuous on $[0,1]$, since there is a jump discontinuity at $x = 1$.

[2.7.16] Problem

Let X be a topological space, and let $A \subseteq X$. Let $f : A \to Y$ be a continuous function to some target codomain Y that is Hausdorff. Show that if f can be uniquely extended to some continuous function $g : \overline{A} \to Y$, then g is uniquely determined by f.

Proof. Suppose $g_1, g_2 : \overline{A} \to Y$ are continuous functions such that $f = g_1|_A = g_2|_A$. For the sake of contradiction, suppose $g_1 \ne g_2$, so there is some $x \in \overline{A} \setminus A$ such that $g_1(x) \ne g_2(x)$. By Hausdorffness of Y, draw disjoint open neighborhoods $V_1, V_2 \subseteq Y$ of $g_1(x), g_2(x)$ respectively. By continuity of f, note $g_1^{-1}(V_1)$ and $g_2^{-1}(V_2)$ are open subsets of X that contain x. As such, $g_1^{-1}(V_1) \cap g_2^{-1}(V_2)$ is also an open neighborhood of x. Since x is a limit point of A, there is some $x_0 \in (g_1^{-1}(V_1) \cap g_2^{-1}(V_2)) \cap (A \setminus \{x\})$. Since $x_0 \in A$, it follows that $f(x_0) = g_1(x_0) = g_2(x_0)$. But $g_1(x_0) \in V_1$ and $g_2(x_0) \in V_2$, which is a contradiction. Thus $g_1 \equiv g_2$, completing the proof.

2.8 Box and Product Topology

We have a notion of Cartesian product for two sets X, Y. Namely, $X \times Y = \{(x, y) : x \in X, y \in Y\}$. But what about the Cartesian product of three sets? Four? Five? Countably many? To tackle this, we redefine the Cartesian product in terms of an arbitrary indexing set.

[2.8.1] Definition (*Cartesian Tuple, Cartesian Product*)

Let Λ be an indexing set. Given a set X, we define a **Λ-tuple** of X to be some function $\mathbf{x} : \Lambda \to X$. We define $x(\lambda)$, more commonly denoted as $\mathbf{x}_\lambda$, as the **λ-th component** of $\mathbf{x}$. We often write $(x_\lambda)_{\lambda\in\Lambda}$ in lieu of $\mathbf{x}$. We denote the set of all Λ-tuples on X by X^Λ, the set of all functions from Λ to X.

Now let $\{A_\lambda\}_{\lambda\in\Lambda}$ be an indexed family of set. We define the **Cartesian product** of this family of sets as the set of all Λ-tuples $(x_\lambda)_{\lambda\in\Lambda}$ such that each $x_\lambda \in A_\lambda$ (that is, the λ-th coordinate of $\mathbf{x}$ lies in A_λ). That is,

$$\text{Cartesian product of } \{A_\lambda\}_{\lambda\in\Lambda} = \prod_{\lambda\in\Lambda} A_\lambda = \left\{ x : \Lambda \to \bigcup_{\lambda\in\Lambda} A_\lambda : x_\lambda \in A_\lambda \,\forall \lambda \in \Lambda \right\}.$$

From here, we can extend the product topology of two sets in different ways. Recall how the basis and subbasis were defined for the product topology on $X \times Y$. We extend these notions to general indexing sets.

[2.8.2] Definition (*Box Topology, Product Topology*)

Suppose $\{X_\lambda\}_{\lambda\in\Lambda}$ is an indexed family of sets.

We define the **box topology** to be the topology generated by the basis

$$\mathcal{B} = \left\{ \prod_{\lambda\in\Lambda} U_\lambda : U_\lambda \text{ is open in } X_\lambda \right\}.$$

We define the **product topology** to be the topology generated by the subbasis

$$\mathcal{S}_\lambda = \{\pi_\lambda^{-1}(U_\lambda) : U_\lambda \text{ is open in } X_\lambda\} \qquad \mathcal{S} = \bigcup_{\lambda\in\Lambda} \mathcal{S}_\lambda.$$

Of course, if $\Lambda = \{1, 2\}$, then these are identical to our previous notion of the product topology on the Cartesian product of two sets. In fact, as we will realize soon, the box and product topology coincide for Cartesian products that come from finite indexing sets.

[2.8.3] Remark (*Comparison Between Box and Product Topology*)

We will compare these topologies by noticing the basis generated by the subbasis for the product topology, then comparing this basis with the basis for the box topology. From the previous definition, let $\{X_\lambda\}_{\lambda\in\Lambda}$ be a family of indexed sets, and define $\mathcal{S}_\lambda$ and $\mathcal{S}$ accordingly.

Recall that a basis is formed by finite intersection of elements of a subbasis. Let's first discover what the finite intersection of elements of $\mathcal{S}_\lambda$ do. Let $O_1,\dots,O_n$ be open subsets of X_λ, then note

$$\bigcap_{k=1}^{n}\pi_\lambda^{-1}(O_k)=\pi_\lambda^{-1}\underbrace{\left(\bigcap_{k=1}^{n}O_k\right)}_{\text{open in } X_\lambda},$$

which is another element of S_λ. Thus, a general basis element $B\in\mathcal{B}$ can be written as

$$B=\bigcap_{k=1}^{n}\pi_{\lambda_k}^{-1}(O_{\lambda_k}),$$

where $\pi_{\lambda_k}:\prod_{\lambda\in\Lambda}X_\lambda\to X_{\lambda_k}$ is the projection map on X_{λ_k}, and O_{λ_k} is an arbitrary open subset of X_{λ_k}. This implies an element $\mathbf{x}=(x_\lambda)_{\lambda\in\Lambda}\in\prod_{\lambda\in\Lambda}X_\lambda$ belongs to some basis B if $x_{\lambda_k}\in O_{\lambda_k}$ for each $k\in\{1,2,\dots,n\}$. Thus, we write B as a product

$$B=\prod_{\lambda\in\Lambda}U_\lambda,$$

where $U_\lambda\subseteq X_\lambda$ is open for $\lambda\in\{1,2,\dots,n\}$, and $U_\lambda=X_\lambda$ otherwise. Recalling that a basic open set in the box topology is the product of any open subset of X_λ for *each* $\lambda\in\Lambda$, not just finitely many of them, we conclude that the box topology is strictly finer than the product topology for an infinite indexing set, and they are otherwise equal for a finite one.

The implication of this is that, for infinite products, an element of a product belongs to a basic open set in the product topology if and only if finitely many of its coordinates lie in the finitely many open sets that generate the basis element. On the contrary, a basic open set in the box topology is determined by $|\Lambda|$ many open sets, not just finitely many.

The standard topology for the Cartesian product is the product topology.

[2.8.4] Theorem

Suppose $\{X_\lambda\}_{\lambda\in\Lambda}$ is an indexed family of sets, and let $X=\prod_{\lambda\in\Lambda}X_\lambda$. Then let $\{\mathcal{B}_\lambda\}_{\lambda\in\Lambda}$ be an indexed family of sets such that $\mathcal{B}_\lambda$ is a basis for each X_λ. Then

$$\mathcal{C}=\prod_{\lambda\in\Lambda}B_\lambda,$$

where $B_\lambda\in\mathcal{B}_\lambda$ for each $\lambda\in\Lambda$, forms a basis for the box topology on X. Alternatively, if $B_\lambda\in\mathcal{B}_\lambda$ for finitely many values of $\lambda\in\Lambda$, and otherwise $B_\lambda=X_\lambda$, then the product forms a basis for the product topology.

Proof. Let $X:=\prod_{\lambda\in\Lambda}X_\lambda$ be endowed with the box topology, and let $O\subseteq X$ be open. We seek to show that, for each $x\in O$, there is some $C\in\mathcal{C}$ such that $x\in C\subseteq O$. If $x\in O$, then by the definition of the box topology, there exists a standard basic open set such that $x\in\prod_{\lambda\in\Lambda}O_\lambda\subseteq O$, where $O_\lambda\subseteq X_\lambda$ is open for each $\lambda\in\Lambda$.

But by definition, because $\mathcal{B}_\lambda$ is a basis for X_λ and the coordinate $x_\lambda\in O_\lambda$, we have that for each $\lambda\in\Lambda$, there exists a basis element $B_\lambda\in\mathcal{B}_\lambda$ such that

$$x_\lambda\in B_\lambda\subseteq O_\lambda.$$

But this implies there is a specific choice of sets to form $C=\prod_{\lambda\in\Lambda}B_\lambda\in\mathcal{C}$ such that

$$x\in\prod_{\lambda\in\Lambda}B_\lambda\subseteq\prod_{\lambda\in\Lambda}O_\lambda\subseteq O,$$

completing the proof that $\mathcal{C}$ is a basis for the box topology. An entirely analogous proof follows for the product topology, with the added condition that $O_\lambda=X_\lambda$ (and thus we can choose $B_\lambda=X_\lambda$) for all but finitely many values of λ. ❦

[2.8.5] Theorem (*Component-Wise Continuity*)

Suppose $\{X_\lambda\}_{\lambda\in\Lambda}$ is a family of indexed sets, where $X = \prod_{\lambda\in\Lambda} X_\lambda$. Suppose $f : A \to X$ is such that $f(a) = (f_\lambda(a))_{\lambda\in\Lambda}$ for every $a \in A$. Then f is continuous if and only if each f_λ is continuous.

Proof.

($\Longrightarrow$): Suppose f is continuous. For any $\lambda \in \Lambda$, note $f_\lambda = \pi_\lambda \circ f$. Note that the product topology forces π_λ to be continuous, f is continuous by hypothesis, and the composition of continuous functions is continuous, so f_λ is continuous.

($\Longleftarrow$): Suppose each f_λ is continuous. To show that f is continuous, it suffices to show that the preimage of every subbasis element of X is open in A. Let S be a standard subbasis element of the product topology on X. Then $S = \pi_\beta^{-1}(U_\beta)$ for some index $\beta \in \Lambda$ and some open set $U_\beta \subseteq X_\beta$. We examine the preimage of S under f:

$$f^{-1}(S) = f^{-1}(\pi_\beta^{-1}(U_\beta)) = (\pi_\beta \circ f)^{-1}(U_\beta) = f_\beta^{-1}(U_\beta).$$

Because f_β is continuous by hypothesis and U_β is open in X_β, the set $f_\beta^{-1}(U_\beta)$ is open in A. Thus, $f^{-1}(S)$ is open in A, meaning f is continuous. ❦

[2.8.6] Remark (*Component-Wise Continuity Fails for Box Topology*)

Define $\mathbb{R}^\omega$ to be the countably infinite product of $\mathbb{R}$ with itself. That is, $\mathbb{R}^\omega = \prod_{k\in\mathbb{Z}_+} \mathbb{R}$. Define the function $f : \mathbb{R} \to \mathbb{R}^\omega$ as the "identity" function $f(t) = (t, t, t, \ldots)$, where $\mathbb{R}$ takes the standard topology and $\mathbb{R}^\omega$ takes the box topology. Note then that $f(t) = (f_n(t))_{n\in\mathbb{Z}_+}$, where each $f_n(t) = t$. Clearly, each f_n is continuous. But is f continuous? Consider the basis element

$$B = (-1, 1) \times \left(-\frac{1}{2}, \frac{1}{2}\right) \times \left(-\frac{1}{3}, \frac{1}{3}\right) \times \cdots,$$

which is open in the box topology. If f is continuous, then $f^{-1}(B)$ must be open. But suppose $x \neq 0$ were in $f^{-1}(B)$—then $f(x) \in B$, meaning $f(x) = (x, x, x, \ldots) \in B$, but there is some $n \in \mathbb{Z}_+$ such that $x \notin (-1/n, 1/n)$, and so $f(x) \notin B$. 0, however, does lie in $f^{-1}(B)$, so this concludes $f^{-1}(B) = \{0\}$, which is not open in $\mathbb{R}$. Thus f is not continuous, emphasizing the previous result only holds for the product topology and *not* the box topology.

2.8.1 Exercises

[2.8.7] Problem

Let $\{X_\lambda\}$ be an indexed product of sets, and let $X = \prod_{\lambda \in \Lambda} X_\lambda$. Let $A_\lambda \subseteq X_\lambda$ be a subspace for each $\lambda \in \Lambda$, and let $A = \prod_{\lambda \in \Lambda} A_\lambda$. Show A is a subspace of X if both products are given the box topology or product topology.

Proof. To show $A \subseteq X$ is a subspace for X, we show that the basis for the box topology on A is the same as the subspace topology. Let B be an arbitrary basis element for the box topology on A, meaning $B = \prod_{\lambda \in \Lambda} O_\lambda$ where $O_\lambda \subseteq A_\lambda$ is open for each $\lambda \in \Lambda$. Since A_λ takes on the subspace topology, it follows that $O_\lambda = U_\lambda \cap A_\lambda$ for some open subset $U_\lambda \subseteq X_\lambda$. Then

$$B = \prod_{\lambda \in \Lambda} (U_\lambda \cap A_\lambda) = \left(\prod_{\lambda \in \Lambda} U_\lambda \right) \cap \left(\prod_{\lambda \in \Lambda} A_\lambda \right) = O \cap A,$$

where $O \subseteq X$ is open in the topology of X. Thus $O \cap A$ is open in the subspace topology of A. Conversely, we show that an arbitrary basis element–say, B—for the subspace topology on A is open in the box topology. Write $B = O \cap A$, where O is a basic open subset of X. Then $O = \prod_{\lambda \in \Lambda} O_\lambda$, where each $O_\lambda \subseteq X_\lambda$ is open, meaning we may write

$$B = \left(\prod_{\lambda \in \Lambda} O_\lambda \right) \cap \left(\prod_{\lambda \in \Lambda} A_\lambda \right) = \prod_{\lambda \in \Lambda} \underbrace{(O_\lambda \cap A_\lambda)}_{\text{open in } A_\lambda},$$

which is open in the box topology of A (by virtue of being a basis element). Thus the subspace and box topologies are equivalent for A, making A a subspace of X.

The proof for the product topology is entirely analogous. ❦

2.9 Metric Topology

[2.9.1] Definition (*Metric, Distance, ε-ball*)

A **metric** on some set X is a function $d : X \times X \to \mathbb{R}$ satisfying the following properties.

(1) **Nonnegativity.** $d(x, y) \geq 0$ for every $x, y \in X$.

(2) **Symmetry.** $d(x, y) = d(y, x)$ for every $x, y \in X$.

(3) **Triangle Inequality.** $d(x, y) + d(y, z) \geq d(x, z)$ for every $x, y, z \in X$.

We say $d(x, y)$ is the **distance** between two points $x, y \in X$. For some $\varepsilon > 0$ and $x_0 \in X$, we define an ε**-ball** about x_0 to be the set of all points a distance less than ε away from x_0. That is,

$$B_\varepsilon(x_0) = \{x \in X : d(x, x_0) < \varepsilon\}.$$

A topological space X is said to be **metrizable** if there exists some metric that induces the topology on X.

[2.9.2] Definition (*Metric Topology*)

Suppose (X, d) is a metric space. Then the set of all ε-balls about every point in X, given by

$$\mathcal{B} = \{B_\varepsilon(x) : \varepsilon > 0, x \in X\}$$

forms a basis for the **metric topology** on X.

Proof. We verify this forms a basis for X.

(1) **Covering.** For any $x \in X$, note $B_1(x)$ is in the basis, and so $X \subseteq \bigcup_{B \in \mathcal{B}} B$ as desired.

(2) **Local Refinement.** Let B_1, B_2 be arbitrary elements from the basis, and let $x \in B_1 \cap B_2$. Thus there must be some δ_1, δ_2 such that $B_{\delta_1}(x) \subseteq B_1$ and $B_{\delta_2}(x) \subseteq B_2$. Then note $B_{\min\{\delta_1, \delta_2\}}(x) \subseteq B_1 \cap B_2$, which is also a basic element, completing the proof.

❦

[2.9.3] Example (*Metric Topology Generates Discrete Topology*)

Given a set X, we may define a metric

$$d : X \times X \to \mathbb{R} \quad d(x, y) = \begin{cases} 1 & x \neq y, \\ 0 & \text{otherwise.} \end{cases}$$

It is obvious d is a metric. But note that the basis for the metric topology contains all singletons, since $\{x\} = B_1(x)$, so the metric topology is discrete.

[2.9.4] Example (*Metric Topology on $\mathbb{R}$ Generates Order Topology*)

Define the standard metric $d(x, y) = |x - y|$ on $\mathbb{R}$. Then note $(a, b) = B_\varepsilon(a + (b - a)/2)$, where $\varepsilon = (b - a)/2$. Similarly, all $B_\varepsilon(x) = (x - \varepsilon, x + \varepsilon)$. Thus the metric topology coincides with the standard order topology on $\mathbb{R}$.

[2.9.5] Definition (*Bounded Subset, Diameter*)

For some metric space (X,d), a set $A \subseteq X$ is said to be **bounded** if there is some $M \in \mathbb{R}$ such that $d(a_1,a_2) \le M$ for every pair $a_1, a_2 \in A$. The **diameter** of a bounded subset A is defined to be

$$\operatorname{diam}(A) = \sup\{d(a_1,a_2) : a_1, a_2 \in A\},$$

which exists since A is bounded above.

Topology is concerned with the idea of metrizability, the abstract notion that a metric may be endowed on a set to form a topology. The behavior and choice of a metric, however, is a strictly analytical topic. Thus, ideas such as "boundedness" that are dependent on choice of metric are not particularly topological. In fact, the next theorem shows that a different metric on $\mathbb{R}$ induces the same topology as the standard metric, making them indistinguishable topologically.

[2.9.6] Theorem (*Standard Bounded Metric Induces Same Topology as Standard Metric*)

Let (X,d) be a metric space. We define the **standard bounded metric** by

$$\overline{d} : X \times X \to \mathbb{R} \quad \overline{d}(x,y) = \min\{d(x,y), 1\}.$$

Then $\overline{d}$ is a metric that induces the same metric topology as d.

Proof. The proof that $\overline{d}$ is a metric is trivial. Let $\mathcal{T}_d, \mathcal{T}_{\overline{d}}$ be the topologies generated by d and $\overline{d}$ respectively. It is obvious that $\mathcal{T}_{\overline{d}} \subseteq \mathcal{T}_d$, so we prove that $\mathcal{T}_d \subseteq \mathcal{T}_{\overline{d}}$. Let $B \in \mathcal{T}_d$. By definition of $\mathcal{T}_d$, for each $x \in B$, we can choose a basis ball about x contained in B; if its radius is 1 or greater, we can simply shrink it. So let B_x be a ball about x such that its radius is less than 1 and $x \in B_x \subseteq B$. Since its radius is less than 1, the ball B_x is identical under d and $\overline{d}$, meaning it is a basic open set of $\mathcal{T}_{\overline{d}}$. Then $B = \bigcup_{x \in B} B_x$, a union of basic open sets of $\mathcal{T}_{\overline{d}}$, completing the proof. ❦

[2.9.7] Example (*Metrizability of $\mathbb{R}^n$*)

Given $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$, we define $\|x\| = \sqrt{x_1^2 + \cdots + x_n^2}$ and define the **Euclidean metric**

$$d(x,y) = \|x - y\|,$$

the standard notion of distance between two points in $\mathbb{R}^n$. We can also define the **square metric** by

$$\rho(x,y) = \max\{|x_1 - y_1|, \ldots, |x_n - y_n|\}.$$

These are intuitive generalizations from the standard metric on $\mathbb{R}$. Indeed, the Euclidean and sqaure metric collapse to the standard metric in $\mathbb{R}^1$. Otherwise, consider basis elements for the Euclidean metric to be balls, whereas basis elements for the square metric are boxes.

[2.9.8] Theorem (*Criterion for Finer Metric Topologies*)

Let X be a set with two metrics d, d', and let $\mathcal{T}, \mathcal{T}'$ be the topologies induced by these metrics respectively. Then $\mathcal{T} \subseteq \mathcal{T}'$ if and only if, for each $x \in X$ and any $\varepsilon > 0$, there is some $\delta > 0$ such that $B_{d'}(x,\delta) \subseteq B_d(x,\varepsilon)$.

Proof.

($\Longrightarrow$): Suppose $\mathcal{T} \subseteq \mathcal{T}'$. Then, for any $x \in X$ and $\varepsilon > 0$, we have that $B_d(x,\varepsilon) \in \mathcal{T}$. Since $\mathcal{T} \subseteq \mathcal{T}'$, $B_d(x,\varepsilon) \in \mathcal{T}'$. Thus, for $x \in B_d(x,\varepsilon)$, by definition of the basis, there is some $\delta > 0$ such that $x \in B_{d'}(x,\delta) \subseteq B_d(x,\varepsilon)$.

($\Longleftarrow$): Let $O \in \mathcal{T}$—we seek to show $O \in \mathcal{T}'$. For each $x \in O$, note $x \in B_d(x,\varepsilon) \subseteq O$ for some $\varepsilon > 0$. By hypothesis, there is some $\delta > 0$ such that $B_x = B_{d'}(x,\delta)$ is such that $x \in B_x \subseteq B_d(x,\varepsilon) \subseteq O$. Then $O = \bigcup_{x \in O} B_x \in \mathcal{T}'$. ❦

[2.9.9] Theorem

The topologies on $\mathbb{R}^n$ induced by the Euclidean metric d and the square metric ρ are identical to the product topology on $\mathbb{R}^n$.

Proof. We begin by showing the Euclidean and square metrics generate the same topologies on $\mathbb{R}^n$. Recalling that basis elements for the Euclidean metric (resp. square metric) topology take the form of balls (resp. congruent boxes), we seek to show that a box can be contained in a sphere and vice versa. For starters, fix the points $x, y \in \mathbb{R}^n$, and let $\vec{y}$ denote the vector whose tip is located at x and tail is located at y. The magnitude of $\vec{y}$, representing the Euclidean distance $d(x, y)$, is bounded below by its largest single coordinate projection $\rho(x, y)$, meaning the box spanned by $\vec{y}$ easily fits within a sphere of radius $d(x, y)$. Conversely, if we bound $\vec{y}$ inside an n-dimensional cube centered at x whose faces are at a distance of $\rho(x, y)$, the Pythagorean theorem guarantees its magnitude cannot exceed the distance to the cube's furthest corner, yielding the upper bound $d(x, y) \leq \sqrt{n}\rho(x, y)$. The statement, which can also be proven with basic algebra, is that

$$\rho(x, y) \leq d(x, y) \leq \sqrt{n}\rho(x, y).$$

In any case, this implies $B_d(x, \varepsilon) \subseteq B_\rho(x, \varepsilon)$ for any $x \in X, \varepsilon > 0$, and same for $B_\rho(x, \varepsilon/\sqrt{n}) \subseteq B_d(x, \varepsilon)$. Thus the topologies are finer than eachother, making them equal.

It suffices to prove that the square metric generates the product topology. Let $\mathcal{T}_\rho$ be the topology generated by the square metric, and let $\mathcal{T}_\times$ be the product topology on $\mathbb{R}^n$. First, let $O \in \mathcal{T}_\rho$. For each $x \in O$, there exists some $\varepsilon > 0$ such that $x \in B_\rho(x, \varepsilon) \subseteq O$. But now note there is some box centered about x whose sidelengths are, at most, ε—this is spanned by basis elements of the product topology, and so $x \in (\text{the box}) \subseteq B_\rho(x, \varepsilon)$. Thus $\mathcal{T}_\rho \subseteq \mathcal{T}_\times$.

Now suppose $O \in \mathcal{T}_\times$. For each $x \in O$, there is some $B = (a_1, b_1) \times \cdots \times (a_n, b_n)$ centered about x such that $x \in B \subseteq O$. With $\varepsilon = \min\{|a_i - b_i| : 1 \leq i \leq n\}$, we note $x \in B_\rho(x, \varepsilon/2) \subseteq B$, completing the proof. ❦

Now we seek to extend these metrices to an infinite product of $\mathbb{R}$. The natural generalizations on a set like $\mathbb{R}^\omega$ are as follows:

$$d(x, y) = \left[\sum_{i=1}^{\infty} (x_i - y_i)^2\right]^{1/2}$$
$$\rho(x, y) = \sup\{|x_n - y_n|\}.$$

But note $d(x, y)$ and $\rho(x, y)$ may not be well-defined due to convergence and boundedness issues. It is, however, possible to generalize a metric on $\mathbb{R}^\omega$, and on $\mathbb{R}^\Lambda$ for some general indexing set Λ. Recall the standard bounded metric $\overline{d}(x, y) = \min\{|x - y|, 1\}$—we may generalize it to $\mathbb{R}^\Lambda$ for some $(x_\lambda)_{\lambda\in\Lambda}, (y_\lambda)_{\lambda\in\Lambda}$ as such:

$$\overline{\rho}(x, y) = \sup_{\lambda\in\Lambda}\{\overline{d}(x_\lambda, y_\lambda)\}.$$

We refer to this as the **uniform metric** on $\mathbb{R}^\Lambda$, and it fittingly generates the **uniform topology**. This has implications for the idea of *uniform convergence*, which we will discuss soon.

[2.9.10] Theorem

For some indexing set Λ, the uniform topology on $\mathbb{R}^\Lambda$ is finer than the product topology and coarser than the box topology. That is, $\mathcal{T}_{\text{product}} \subseteq \mathcal{T}_{\text{uniform}} \subseteq \mathcal{T}_{\text{box}}$.

Proof. Let $\mathbf{x} = (x_\lambda)_{\lambda\in\Lambda} \in \mathbb{R}^\Lambda$. Let $U = \prod_{\lambda\in\Lambda} U_\lambda$ be an element of the product topology's basis about $\mathbf{x}$, where $\lambda_1, \ldots, \lambda_n$ are the indices for which $U_\lambda \neq \mathbb{R}$.

Let $z \in U$. We seek to show there is some $\varepsilon > 0$ such that $z \in B_{\overline{\rho}}(x, \varepsilon) \subseteq U$. For each λ_i, choose $\varepsilon_i > 0$ such that $B_{\overline{d}}(x, \varepsilon_i) \subseteq U_{\lambda_i}$. Then let $\varepsilon = \min\{\varepsilon_1, \ldots, \varepsilon_n\}$.

On the other hand, let $x \in B_{\overline{\rho}}(x, \varepsilon)$ for some $\varepsilon > 0$. For each $\lambda \in \Lambda$, let $X_\lambda = (x_\lambda - \varepsilon/2, x_\lambda + \varepsilon/2)$, then note $x \in \prod_{\lambda\in\Lambda} X_\lambda \subseteq B_{\overline{\rho}}(x, \varepsilon)$. ❦

[2.9.11] Theorem (*Continuity For Functions Between Metric Spaces*)

Let $f : X \to Y$ be a map between two metric spaces (X, d_X) and (Y, d_Y). Then f is continuous if and only if, for any $x \in X$ and $\varepsilon > 0$, there is some $\delta > 0$ such that

$$d_X(x, a) < \delta \implies d_Y(f(x), f(a)) < \varepsilon.$$

Proof.

($\Longrightarrow$): Suppose f is continuous. Fix $x \in X$ and $\varepsilon > 0$, and let $O = B_{d_Y}(f(x), \varepsilon)$. Since O is open, it follows that $f^{-1}(O)$ is open. Since $f^{-1}(O)$ is the union of basic, open sets in X, there is some $\delta > 0$ such that $B_{d_X}(x, \delta) \subseteq f^{-1}(O)$. Then, for any $a \in B_{d_X}(x, \delta)$, it follows that $f(a) \in O$.

($\Longleftarrow$): Suppose that, for any $x \in X$ and $\varepsilon > 0$, there is some $\delta > 0$ such that

$$d_X(x, a) < \delta \implies d_Y(f(x), f(a)) < \varepsilon.$$

Let O be open in Y—we show $f^{-1}(O)$ is open in X. Let $x \in f^{-1}(O)$. Let $\varepsilon > 0$ be such that $B_{d_Y}(f(x), \varepsilon) \subseteq O$. By hypothesis, there is some $\delta > 0$ such that $B_x = B_{d_X}(x, \delta)$, where $B_x \subseteq f^{-1}(O)$. Then $f^{-1}(O) = \bigcup_{x \in f^{-1}(O)} B_x$, which is open. ❦

[2.9.12] Theorem (*Sequence Lemma*)

Let X be a topological space, and let $A \subseteq X$. If there is a sequence $(x_n) \subseteq A$ such that $x_n \to x$, then $x \in \overline{A}$. Conversely, if X is metrizable and $x \in \overline{A}$, there is some sequence of points $(x_n) \subseteq A$ such that $x_n \to x$.

Proof.

($\Longrightarrow$): Suppose $(x_n) \subseteq A$ is such that $x_n \to x$. Then, for any open neighborhood O of x, there is some $N \in \mathbb{Z}_+$ such that $x_n \in O$ for every $n \geq N$. Thus $O \cap A \neq \emptyset$ and so $x \in \overline{A}$.

($\Longleftarrow$): Suppose X has some metric d. Fix $x \in \overline{A}$, and fix some arbitrary open neighborhood O of x. By definition of the closure, $O \cap A \neq \emptyset$. Since O is open in a metric space, there is some $\varepsilon > 0$ such that $B(x, \varepsilon) \subseteq O$ and $B(x, \varepsilon) \cap A \neq \emptyset$. Then there is some $N \in \mathbb{Z}_+$ such that $1/n < \varepsilon$ for every $n \geq N$. Let $x_n = a$ for some $a \in A$ and all $n < N$, and otherwise let x_n be some arbitrary element of $B(x, 1/n) \cap A$, which is guaranteed to exist. Then $(x_n) \subseteq A$ and $x_n \to x$. ❦

[2.9.13] Theorem (*Characterization of Continuity*)

Let $f : X \to Y$. If f is continuous, then for every $x_n \to x$ in X, it follows that $f(x_n) \to f(x)$. Conversely, if X is metrizable, we have that if every $x_n \to x$ in X is such that $f(x_n) \to f(x)$, then f is continuous.

Proof.

($\Longrightarrow$): Suppose f is continuous. Let $(x_n) \subseteq X$ be a sequence such that $x_n \to x$. Let O be an open neighborhood of $f(x)$. Then note $f^{-1}(O)$ is open by f's continuity, and $x \in f^{-1}(O)$. Thus there is some $N \in \mathbb{Z}_+$ such that $x_n \in f^{-1}(O)$ for every $n \geq N$. Thus $f(x_n) \in O$ for every $n \geq N$, meaning $f(x_n) \to f(x)$.

($\Longleftarrow$): Suppose X has some metric d. Suppose that, for every $(x_n) \subseteq X$ such that $x_n \to x$, it follows that $f(x_n) \to f(x)$. Let $A \subseteq X$—we will show f is continuous by proving $f(\overline{A}) \subseteq \overline{f(A)}$. For starters, suppose $x \in \overline{A}$. Since X is metrizable, A inherits a metric subspace topology, and by the sequence lemma there is some $(x_n) \subseteq A$ such that $x_n \to x$. By hypothesis, we have $f(x_n) \to f(x)$, meaning $f(x) \in \overline{f(A)}$. Thus $f(\overline{A}) \subseteq \overline{f(A)}$, completing the proof. ❦

[2.9.14] Definition (*Uniform Convergence*)

Let $f_n : X \to Y$ be a sequence of functions that map the set X into the metric space Y. We say $(f_n) \to f$ converges **uniformly** if, for any $\varepsilon > 0$, there is some $N \in \mathbb{Z}_+$ such that

$$d(f_n(x), f(x)) < \varepsilon$$

for every $n \geq N$ and every $x \in X$.

This mode of convergence is subtly distinct from traditional, pointwise convergence. Pointwise convergence only mandates that $(f_n(x)) \to f(x)$ for every $x \in X$ for large enough n, but this choice of "large enough" n can vary based on choice of x. If our choice of large enough n cannot be made independent from our choice of x, then the convergence is not uniform. Pointwise convergence is quite weak. We will soon see that if $(f_n) \to f$ with each f_n continuous, f is not necessarily continuous unless we impose extra hypotheses, which is where the uniformity of the convergence comes in.

[2.9.15] Theorem (*Uniform Limit Theorem*)

Suppose $f_n : X \to Y$ is a sequence of continuous functions that map the set X into the metric space Y. If $(f_n) \to f$ is uniform, then f is continuous.

Proof. Let $x \in X$ and $\varepsilon > 0$. We need to find $\delta > 0$ such that

$$d(x, y) < \delta \implies d(f(x), f(y)) < \varepsilon.$$

Since $f_n \to f$ is uniform, there is some $N \in \mathbb{Z}_+$ such that $d(f(x), f_N(x)) < \varepsilon/3$ for every $x \in X$. Then there is some $\delta > 0$ such that

$$d(x, y) < \delta \implies d(f_N(x), f_N(y)) < \varepsilon/3,$$

by continuity of each f_n. Then, if $d(x, y) < \delta$, we have that

$$\begin{aligned} d(f(x), f(y)) &\leq d(f(x), f_N(x)) + d(f_N(x), f_N(y)) + d(f_N(y), f(y)) \\ &= \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon, \end{aligned}$$

completing the proof. ❦

2.9.1 Exercises

[2.9.16] Problem

Let X be a set, and let $f_n : X \to \mathbb{R}$ be a sequence of functions. Let $\overline{\rho}$ be the uniform metric on the space $\mathbb{R}^X$, the set of all functions from X to $\mathbb{R}$. Show that $f_n \to f$ uniformly if and only if $f_n \to f$ as elements of the metric space $(\mathbb{R}^X, \overline{\rho})$.

Proof.

($\Longrightarrow$): Suppose $f_n \to f$ uniformly. To show $f_n \to f$ in $(\mathbb{R}^X, \overline{\rho})$, we choose some arbitrary open neighborhood O of f and show there is some $N \in \mathbb{Z}_+$ such that $f_n \in O$ for every $n \geq N$. Note there is some $\varepsilon > 0$ such that $f \in B(f, \varepsilon) \subseteq O$. By $f_n \to f$ being uniform, there is some $N_1 \in \mathbb{Z}_+$ such that $|f_n(x) - f(x)| < 1$ for every $n \geq N_1$. Then

$$\overline{\rho}(f_n, f) = \sup_{x \in X}\{\overline{d}(f_n(x), f(x))\} = \sup_{x \in X}\{|f_n(x) - f(x)|\},$$

with the second equality coming from the aforementioned fact. Again, by uniformity of $f_n \to f$, there is some $N > N_1 \in \mathbb{Z}_+$ such that $\overline{\rho}(f_n, f) \leq \varepsilon/2$ for every $n \geq N$. Thus $f_n \in B(f, \varepsilon) \subseteq O$ for every $n \geq N$, meaning $f_n \to f$ as elements of $\mathbb{R}^X$ as desired.

($\Longleftarrow$): Suppose $f_n \to f$ as elements of $\mathbb{R}^X$. Fix $\varepsilon > 0$. We seek to show there is some $N \in \mathbb{Z}_+$ such that

$$d(f_n(x), f(x)) < \varepsilon$$

for every $x \in X$ and $n \geq N$. Let $\varepsilon' = \min\{\varepsilon/2, 0.5\}$. Then let $O = B(f, \varepsilon')$ be an open set in $\mathbb{R}^X$. There is some $N \in \mathbb{Z}_+$ such that $f_n \in O$ for each $n \geq N$ by hypothesis. For any $n \geq N$, note that $\overline{\rho}(f_n, f) < \varepsilon' < 1$. Thus $\overline{d}(f_n(x), f(x)) < \varepsilon' < 1$ for every $x \in X$, and so $\overline{d}(f_n(x), f(x)) = d(f_n(x), f(x)) < \varepsilon' < \varepsilon$ for every $x \in X$ and $n \geq N$, meaning $(f_n) \to f$ uniformly. ❦

2.10 Quotient Topology

[2.10.1] Definition (*Quotient Map*)

Let X, Y be topological spaces, and let $p : X \to Y$ be surjective. Then p is a **quotient map** if every subset $U \subseteq Y$ is open if and only if $p^{-1}(U)$ is open.

[2.10.2] Remark (*Stronger than Continuity, Weaker than Homeomorphism*)

A quotient map must be surjective, because then $p^{-1}(\emptyset) = \emptyset$. If $p^{-1}(\emptyset)$ weren't empty, then we can force a non-open set to be the preimage of the open set, contradicting the definition.

Note this is stronger than continuity. p would be continuous if $p^{-1}(U) \subseteq X$ was open given that $U \subseteq Y$ is open, which is only the reverse direction for the definition of a quotient map. For p to be a quotient map, it must be continuous, surjective, and it must be true that if $p^{-1}(U) \subseteq X$ is open, then $U \subseteq Y$ is too.

Note that a quotient map does **not** necessarily take an open $O \subseteq X$ to an open $p(O) \subseteq Y$. By definition, we'd need $p^{-1}(p(O))$ to be open, but $p^{-1}(p(O)) \subseteq O$, and subsets of O need not be open. $p^{-1}(p(O)) = O$ is forced when p is injective, but this would make p a bijective map that is continuous both ways, making p a homeomorphism. Indeed, a quotient map is simply a less restrictive homeomorphism, one that gets rid of injectivity in the hypothesis.

[2.10.3] Definition (*Saturated Set*)

A set $C \subseteq X$ is ***saturated*** *with respect to* f if it has the property that if it intersects some fiber $f^{-1}(\{y\})$, then C contains the entire fiber. That is, $C \subseteq X$ is saturated if it is the union of fibers of f.

We can reformulate a quotient map to be a map that takes open (resp. closed) saturated subsets of X to open (resp. closed) subsets of Y.

[2.10.4] Definition (*Open Map, Closed Map*)

Let $f : X \to Y$ be a map between two topological spaces. f is an **open map** if, for every open $U \subseteq X$, $f(U)$ is open. Analogously, f is a **closed map** if, for every closed $U \subseteq X$, $f(U)$ is closed.

Immediately from definition, it follows that surjective, continuous maps that are either open or closed are quotient maps. For example, the projection map $\pi : X \times Y \to X$ is surjective, continuous, and maps open sets to open sets, making it a quotient map.

[2.10.5] Definition (*Quotient Topology*)

Let X be a topological space and let A be a set. Let $p : X \to A$ be a surjective map. Then there is exactly one topology, the **quotient topology**, that can be endowed on A to make p a quotient map.

[2.10.6] Remark (*Verification of Quotient Topology*)

The definition of a quotient map makes the quotient topology unique. Indeed, with our previous definition in mind, our topology on A must be such that U is in the topology if and only if $p^{-1}(U)$ is open in X. Of course, the sets $\emptyset$ and A are open in the quotient topology, since $p^{-1}(\emptyset) = \emptyset$ and $p^{-1}(A) = X$, which are open sets in X. The conditions on arbitrary unions and finite intersections come by how preimages work:

$$p^{-1}\left(\bigcup_{\lambda\in\Lambda} U_\lambda\right) = \bigcup_{\lambda\in\Lambda} p^{-1}(U_\lambda), \qquad \text{[each } U_\lambda \text{ open in the quotient topology]}$$

$$p^{-1}\left(\bigcap_{k=1}^{n} U_k\right) = \bigcap_{k=1}^{n} p^{-1}(U_k), \qquad \text{[each } U_k \text{ open in the quotient topology]}$$

and so the quotient topology is a valid topology. Moreover, it is uniquely determined by the topology on X, since the only subsets of A that are open are the ones such that $p^{-1}(U)$ is open. Adding an additional set to the quotient topology would make p no longer continuous, whereas omitting an existing set in the quotient topology would make p no longer a quotient map.

[2.10.7] Definition (*Quotient Space*)

Let X be a topological space, and let X^* be a partition of X (a set of disjoint subsets of X whose union is X). Let $p : X \to X^*$ be an identification map that takes elements of X to the subset they belong to in X^*. In the quotient topology induced by p, X^* is called a **quotient space** of X.

[2.10.8] Remark (*Interlude on Equivalence Classes*)

Recall the idea behind the canonical set decomposition in Set and its representation as the **First Isomorphism Theorem** in Grp.

For any set X and some map out of it, say $f : X \to Y$, the image of the map induces an equivalence relation that partitions X. Call this partition X^*. We define $\ker(f) = \sim$ to be such that $x_1 \sim x_2 \iff f(x_1) = f(x_2)$. Then two elements of X belong to the same subset of X^* if their image is the same under f. Essentially, X^* changes the notion of equivalence, where two points of X are indistinguishable in X^* if they are "equivalent" to eachother, where equivalence is defined by having the same image under f.

We can decompose the action of f into three steps. First, we can define a map $\pi : X \to X^*$ that sends an element in X to its equivalence class in X^*. Then, since X^* is defined specifically so that elements of its equivalence classes have the same image, we can define a map $\tilde{f} : X^* \to \mathrm{im}(f)$ that sends each equivalence class to its image. Finally, we can define an inclusion map $\iota : \mathrm{im}(f) \to Y$ that embeds the image of f into its codomain. This is represented by this commutative diagram.

$$\begin{tikzcd} A \arrow[r, two heads, "\pi"'] \arrow[rrr, bend left, "f"] & A/\sim \arrow[r, "\tilde{f}"'] & \mathrm{im}(f) \arrow[r, hook, "i"'] & B \end{tikzcd}$$

Notice that $\tilde{f}$ is a bijection. Equating points of X whose image is equal under f resolves the non-injectivity of f, since the many points that may map to an element in the image collapse to one point under $\sim$. Mapping into $\mathrm{im}(f)$ instead of Y automatically resolves the non-surjectivity of f. If f is either injective or surjective, then these processes collapse trivially—namely, if f is injective, then X^* is a set of singleton subsets of X, and if f is surjective, then $\mathrm{im}(f) = Y$.

[2.10.9] Remark (*Reformulation of Open Sets in Quotient Space*)

Let X be a topological space and let X^* be a partition of X that is a quotient space relative to the identification map $p : X \to X^*$. Recall a set U is open in the quotient topology endowed on X^* is $p^{-1}(U)$ is open. A subset $U \subseteq X^*$ is a set of equivalence classes of X, and its preimage under p is simply the union of each equivalence class. Thus U is open if the union of equivalence classes it contains is an open subset of X.

Now we observe how the quotient map behaves in the context of subspaces, products, and other constructions. First, we note that the restriction of a quotient map to a subspace is not necessarily a quotient map. We will consider extra hypotheses that do let the restriction of a quotient map be a quotient map.

[2.10.10] Theorem (*Restriction of Quotient Map*)

Suppose $p : X \to Y$ is a quotient map, and let $A \subseteq X$ be a subspace. Then let $p|_A = q : A \to p(A)$ be the restriction.

(1) If A is an open or closed subset of X, then q is a quotient map.

(2) If p is an open or closed map, then q is a quotient map.