1 Set Theory and Logic

1.3 Relations

[1.3.1] Definition (*Relation*)

A relation on a set A is some subset $C \subseteq A \times A$. We say that $x, y \in A$ are **related** under C, denoted xCy, if $(x, y) \in C$.

[1.3.2] Example (*Function as Relation*)

Note a function $f : A \to A$ determines a relation

$$\Gamma_f = \{(a, f(a)) : a \in A\} \subseteq A \times A.$$

This relation is unique in that each $a \in A$ is related to precisely one element, namely $f(a)$.

[1.3.3] Definition (*Equivalence Relation*)

A relation $\sim$ on a set A is said to be an **equivalence relation** if the following hold true.

- **Reflexivity.** $x \sim x$ for every $x \in A$.
- **Symmetry.** $x \sim y \iff y \sim x$ for every $x, y \in A$.
- **Transitivity.** $x \sim y$ and $y \sim z \implies x \sim z$ for every $x, y, z \in A$.

[1.3.4] Definition (*Equivalence Class*)

For a set A equipped with an equivalence relation $\sim$, we may write the **equivalence class** of a:

$$[a]_\sim = \{x \in A : a \sim x\}.$$

[1.3.5] Definition (*Partition*)

A partition of some set A is a set $\mathcal{E}$ such that

$$\bigcup_{E \in \mathcal{E}} E = A \qquad \bigcap_{E \in \mathcal{E}} E = \emptyset.$$

That is, $\mathcal{E}$ is a set of disjoint subsets of A that, when combined, form A.

[1.3.6] Theorem (*Equivalence Classes Form a Partition*)

Supose A is a set equipped with an equivalence relation $\sim$. Suppose $E = [x]_\sim$ and $E' = [x']_\sim$ for some $x, x' \in A$. E and E' are either disjoint or equal.

> **Proof.** Suppose $E \cap E' \neq 0$, so there is some $a \in (E \cap E')$. Then $x \sim a$ and $a \sim x'$, so $x \sim x'$. Thus $E \subseteq E'$ (and by symmetry, since $x \sim x'$ means $x' \sim x$), $E' \subseteq E$. Thus $E = E'$. ❦

[1.3.7] Remark (*Equivalence Relations and Partitions*)

As aforementioned, all equivalence relations partition a set. Moreover, a partition is induced by an equivalence relation. Indeed, if $\mathcal{E}$ is a partition of A, we may define $\sim$ such that $x \sim y$ if and only if x and y belong to the same set inside $\mathcal{E}$.

[1.3.8] Definition (*Order Relation*)

A relation C on a set A is called an **order relation** (or a simple/linear order) if it has the following properties.

(1) **Comparability.** If $x, y \in A$ are such that $x \neq y$, then xCy or yCx.

(2) **Nonreflexivity.** For any $x \in A$, xCx is false.

(3) **Transitivity.** For any $x, y, z \in A$, if xCy and yCz, then xCz.

An order relation on a set is typically denoted by $<$.

[1.3.9] Example (*Dictionary Order Relation*)

Suppose A, B are sets equipped with order relations $<_A, <_B$. Define $<$ on $A \times B$ such that, for any $(a_1, b_1), (a_2, b_2) \in A \times B$, it is such that

$$(a_1, b_1) < (a_2, b_2)$$

if $a_1 <_A a_2$, or $a_1 = a_2$ and $b_1 <_B b_2$.

[1.3.10] Definition (*Greatest Lower Bound Property, Least Upper Bound Property*)

An ordered set A is said to have the **least upper bound** property if, for any bounded $A_0 \subseteq A$, the set of upper bounds on A_0 has a least element. The **greatest lower bound** is defined conversely.

1.4 The Integers and Real Numbers

We assume the real numbers exist and satisfy trivial axioms. From the reals we produce the integers.

[1.4.1] Definition (*Inductive Subset, Positive Integers*)

A subset $A \subseteq \mathbb{R}$ is said to be **inductive** if $1 \in A$ and if, for every $x \in A$, the number $x+1 \in A$. Let $\mathcal{A}$ be the collection of all inductive subsets of $\mathbb{R}$. Define the **positive integers** to be the intersection of all elements of $\mathcal{A}$.

$$\mathbb{Z}_+ = \bigcap_{A \in \mathcal{A}} A.$$

Note this implies $\mathbb{Z}_+$ is the smallest inductive subset of $\mathbb{R}$. Note that, of course, $\mathbb{Z}_+$ itself is inductive. Note that if A is any inductive set of positive integers, then it is necessarily $\mathbb{Z}_+$.

We define the set of integers $\mathbb{Z}$ to be the set of positive integers $\mathbb{Z}_+$, 0, and the set of negative integers $-\mathbb{Z}_+$. The set of integers is closed under addition, subtraction, and multiplication, but is not under division—indeed, the set of rationals $\mathbb{Q}$ is said to be the set of all possible quotients of integers.

[1.4.2] Definition (*Section*)

A **section** of the positive integers, denoted S_n, is such that

$$S_{n+1} = \{1, \ldots, n\}$$

for each $n \in \mathbb{Z}_+$.

[1.4.3] Theorem (*Well-Ordering Property*)

Every nonempty subset of $\mathbb{Z}_+$ has a smallest element.

Proof. We first show that for each $n \in \mathbb{Z}_+$, every nonempty subset of the finite segment $\{1, \ldots, n\}$ has a smallest element. For the base case $n = 1$, the only nonempty subset is $\{1\}$ itself, which has 1 as its smallest element. Now, assume the statement holds for n. Let C be a nonempty subset of $\{1, \ldots, n+1\}$. If C consists only of the element $n+1$, then $n+1$ is clearly the smallest element. Otherwise, C must contain elements from $\{1, \ldots, n\}$. In this case, the intersection $C \cap \{1, \ldots, n\}$ is a nonempty subset of $\{1, \ldots, n\}$, so by our inductive hypothesis, it has a smallest element k. Since $k \leq n < n+1$, k is also the smallest element of the entire set C. Thus, by induction, every nonempty subset of any finite segment $\{1, \ldots, n\}$ has a smallest element.

Now let D be any nonempty subset of $\mathbb{Z}_+$. To find its smallest element, choose any element $n \in D$. We only need to look at the elements of D that are less than or equal to n. Let $A = D \cap \{1, \ldots, n\}$. This set A is nonempty and is a subset of a finite segment, so it must have a smallest element k. Because k is the smallest in A, and every element in D that is not in A is strictly greater than n, it follows that k is the smallest element of D. ❦

[1.4.4] Theorem (*Strong Induction Principle*)

Let $A \subseteq \mathbb{Z}_+$. Suppose that for each $n \in \mathbb{Z}_+$, the statement $S_n \subseteq A$ implies $n \in A$. Then $A = \mathbb{Z}_+$.

Proof. Suppose A does not equal $\mathbb{Z}_+$. This means there are some positive integers missing from A. By the Well-Ordering Property, there must be a smallest positive integer n that is not in A. Since n is the smallest element missing, every positive integer less than n must already be in A. This is exactly the condition $S_n \subseteq A$. Our hypothesis states that if $S_n \subseteq A$, then n must be in A. This creates a contradiction: n cannot be both missing from A and required to be in A. Therefore, there can be no missing elements, and $A = \mathbb{Z}_+$. ❦

1.5 Cartesian Products

[1.5.1] Definition (*Indexing Function, Set, and Family*)

Suppose $\mathcal{A}$ is a nonempty collection of sets. An **indexing function** for $\mathcal{A}$ is a surjective function $f : \Lambda \to \mathcal{A}$, where Λ is called the **indexing set**. For each $\lambda \in \Lambda$, we write $A_\lambda = f(\lambda)$. The collection $\{A_\lambda\}_{\lambda \in \Lambda}$ is called the **indexed family of sets** associated with f.

We distinguish the collection $\mathcal{A}$ from the indexed family $\{A_\lambda\}_{\lambda \in \Lambda}$: the latter includes labeling information and may repeat the same set for different indices.

We may use indexing functions to define arbitrary unions and intersections. Suppose $f : \Lambda \to \mathcal{A}$ is an indexing function for some collection of sets $\mathcal{A}$. We may define

$$\bigcup_{\lambda \in \Lambda} A_\lambda = \{x : \text{for at least one } \lambda \in \Lambda, x \in A_\lambda\},$$

$$\bigcap_{\lambda \in \Lambda} A_\lambda = \{x : \text{for every least one } \lambda \in \Lambda, x \in A_\lambda\}.$$

These are, effectively, the unions and intersections of every element of $\mathcal{A}$. For the specific case where $\lambda = \mathbb{Z}_+$, we can denote the indexed family by $\{A_1, \ldots, A_n\}$, while denoting the union and intersection of each A_n by

$$A_1 \cup \cdots \cup A_n \qquad A_1 \cap \cdots \cap A_n.$$

[1.5.2] Definition (*m-Tuple*)

Let $m \in \mathbb{Z}_+$. Given a set X, we define an ***m*-tuple** of the elements of X to be a function

$$\mathbf{x} : \{1, \ldots, m\} \to X.$$

We denote the value of $\mathbf{x}$ at i, $\mathbf{x}(i)$, by x_i and call it the ***i*-th coordinate** of $\mathbf{x}$. We typically denote the entire function by

$$(x_1, \ldots, x_m).$$

[1.5.3] Definition (*Cartesian Product, Revisited*)

Let $\{A_1, \ldots, A_m\}$ be a family of sets indexed by $\{1, \ldots, m\}$. Let $X = A_1 \cup \cdots \cup A_m$. We define the **Cartesian product** of this indexed family, denoted by

$$\prod_{i=1}^{m} A_i \quad \text{or} \quad A_1 \times \cdots \times A_m$$

to be the set of all m-tuples $(x_1, \ldots, x_m)$, where $x_k \in A_k$.

In a similar vein, we define ω-tuples to be m-tuples, simply with their domain equal to $\mathbb{Z}_+$. An ω-tuple is simply an infinite sequence, and the Cartesian product of countably many sets is the set of all ω-tuples, analogous to our definition of Cartesian products of finitely many sets.

1.6 Takeaways

We simply discuss the important ideas of logic and set theory here. In mathematics, there are two parallel ways to define how elements compare to one another: **non-strict** orders (think $\leq$) and **strict** orders (think $<$). We begin by axiomatizing non-strict ordering, discuss the axioms for strict ordering, and highlighting their equivalence.

[1.6.1] Definition (*Partially Ordered Set*)

A set X is said to be a **partially ordered set** (or "poset") if it is equipped with a relation $\leq$ that satisfies the following properties:

(1) **Reflexivity.** For every $x \in X$, $x \leq x$.

(2) **Antisymmetry.** For any $x, y \in X$, if $x \leq y$ and $y \leq x$, then $x = y$.

(3) **Transitivity.** For any $x, y, z \in X$, if $x \leq y$ and $y \leq z$, then $x \leq z$.

[1.6.2] Definition (*Totally Ordered Set*)

A poset X is called a **totally ordered set** (also called a "chain" or "linear order") if every pair of elements is comparable. That is, for any $x, y \in X$, either $x \leq y$ or $y \leq x$.

[1.6.3] Definition (*Strict Partial Order*)

A relation $<$ on a set X is a **strict partial order** if it satisfies:

(1) **Nonreflexivity.** For every $x \in X$, $x < x$ is false.

(2) **Transitivity.** For any $x, y, z \in X$, if $x < y$ and $y < z$, then $x < z$.

[1.6.4] Definition (*Strict Total Order*)

A relation $<$ on a set X is a **strict total order** (often referred to in topology simply as an **order relation**) if it is a strict partial order that also satisfies the law of trichotomy:

3. **Comparability.** For any $x, y \in X$ such that $x \neq y$, either $x < y$ or $y < x$.

[1.6.5] Remark (*The Equivalence of Strict and Non-Strict Orders*)

Strict and non-strict orders are simply two different languages describing the exact same underlying mathematical structure. You can always translate one into the other seamlessly:

From Non-Strict to Strict: Given a non-strict order $\leq$, we induce a strict order $<$ by declaring:

$$x < y \iff x \leq y \text{ and } x \neq y.$$

From Strict to Non-Strict: Given a strict order $<$, we induce a non-strict order $\leq$ by declaring:

$$x \leq y \iff x < y \text{ or } x = y.$$

From now on, we assume partially/totally ordered sets are equipped with strict orders that induce nonstrict orders.

[1.6.6] Definition (*Upper Bound*)

Suppose X is a partially ordered set. For any subset $S \subseteq X$, we say $x \in X$ is an **upper bound** for S if $x \geq s$ for every $s \in S$.

[1.6.7] Definition (*Maximal Element*)

Suppose X is a partially ordered set. For any subset $S \subseteq X$, we say $x \in S$ is a **maximal element** for S if $x \geq s$ for every $s \in S$.

[1.6.8] Axiom (*Zorn's Lemma*)

Suppose X is a non-empty partially ordered set for which every totally ordered subset (i.e., a chain) has an upper bound. Then X has a maximal element.

[1.6.9] Example (*All Vector Spaces Have Bases*)

Essentially, if $X \neq \emptyset$ is a partially ordered set for which every totally ordered subset is bounded above, then X must have one element that is not less than any other element. We cleverly apply this to vector spaces to force the existence of a basis.

Fix V to be an arbitrary vector space, even one that may be infinite-dimensional. Then, we consider the set X of all linearly independent subsets of V. We define a partial ordering on X, such that for any $U, W \in X$, we say

$$U \leq W \iff U \subseteq W.$$

Of course, $X \neq \emptyset$ since the empty set $\emptyset$ is trivially linearly independent, so X is a valid partially ordered set.

To apply Zorn's Lemma, let C be an arbitrary totally ordered subset (a chain) in X. We propose that the union of all sets in C, denoted $M = \bigcup_{U \in C} U$, is an upper bound for C. Clearly, $U \subseteq M$ for all $U \in C$. However, to be a valid upper bound in X, we must verify that $M \in X$; that is, M must be linearly independent.

Suppose for the sake of contradiction that M is linearly dependent. Then, there exists a finite set of vectors $\{v_1, v_2, \ldots, v_n\} \subseteq M$ and scalars $c_1, \ldots, c_n$ (not all zero) such that $c_1 v_1 + \cdots + c_n v_n = 0$. Since M is the union of the chain C, each vector v_i must belong to some set $U_i \in C$. Because C is totally ordered, for any finite collection of sets in C, one of them must contain all the others. Let U_{max} be the largest of these sets $U_1, \ldots, U_n$. This implies that all $v_i \in U_{max}$. But $U_{max} \in X$, meaning it is linearly independent, which contradicts our assumption that these vectors form a linear dependence. Therefore, M must be linearly independent, meaning $M \in X$ and C is indeed bounded above.

By Zorn's Lemma, X contains a maximal element, say B. Because $B \in X$, we already know B is linearly independent. All that remains is to show that B spans V. Suppose it does not; then there exists some vector $v \in V \setminus \mathrm{Span}(B)$. But then the set $B \cup \{v\}$ would be linearly independent, meaning $B \cup \{v\} \in X$. Since $B \subsetneq B \cup \{v\}$, this strictly contradicts the maximality of B. Thus, $\mathrm{Span}(B) = V$, and we conclude that B is a basis for V.

[1.6.10] Axiom (*Axiom of Choice*)

Suppose $\mathcal{C}$ is a collection of non-empty sets. Then you may construct a new set by "choosing" exactly one element from each set.

[1.6.11] Remark

Of course, this seems trivial. If you have infinitely many pairs of shoes, you don't need a special axiom to pick one from each pair (just pick only the left ones). But if you have infinitely many pairs of identical socks, there is no definable rule to pick one from each. The Axiom of Choice simply declares you can do it anyways. It assumes the existence of such a set, even without a definite construction.

[1.6.12] Definition (*Well-Ordered Set*)

A totally ordered set $(X, \leq)$ is said to be **well-ordered** if, for any nonempty subset $A \subseteq X$, there exists a strictly least (minimum) element.

[1.6.13] Axiom (*Well-Ordering Axiom*)

Every set can be equipped with a total ordering such that it is well-ordered.