6 Paracompactness

6.1 Local Finiteness

[6.1.1] Definition (*Local Finiteness*)

Let X be a topological space together with a collection of subsets $\mathcal{A}$. We say $\mathcal{A}$ is **locally finite** if every point $x \in X$ has some open neighborhood U of x such that only finitely many members of $\mathcal{A}$ intersect U.

[6.1.2] Example

Note that $\mathcal{A} = \{(n, n+1)\}_{n \in \mathbb{Z}_+}$ is a locally finite collection with respect to $\mathbb{R}$. Indeed, for any $x \in X$, the open neighborhood $(x - 0.5, x + 0.5)$ is intersected by, at most, three members of $\mathcal{A}$. Also note that $\{(0, 1/n)\}_{n \in \mathbb{Z}_+}$ is a locally finite collection on $(0, 1)$, but not on $\mathbb{R}$ (since 0, 1 are limit points and thus admit countably many elements of this collection into any neighborhood of those points).

[6.1.3] Theorem (*Subset and Closure Properties of Locally Finite Collections*)

Let $\mathcal{A}$ be a locally finite collection of subsets of X.

(1) Any subcollection of $\mathcal{A}$ is locally finite.

(2) The collection $\mathcal{B} = \{\overline{A}\}_{A \in \mathcal{A}}$ is a locally finite collection.

(3) $\overline{\bigcup_{A \in \mathcal{A}} A} = \bigcup_{A \in \mathcal{A}} \overline{A}$.

Proof.

(1): Immediate.

(2): Let $x \in X$ be arbitrary. Choose an open neighborhood U of x such that the only members of $\mathcal{A}$ that intersect U are $\{A_k\}_{k=1}^n$. Let $m \notin \{1, \ldots, n\}$ such that $A_m \in \mathcal{A}$, then note $U \cap \overline{A_m} = \emptyset$ since $U \cap A_m = \emptyset$. Thus U intersects finitely many elements of $\mathcal{B}$.

(3): Let $Y = \bigcup_{A \in \mathcal{A}} A$. It is a standard topological fact that $\bigcup_{A \in \mathcal{A}} \overline{A} \subseteq \overline{Y}$. We prove the reverse inclusion. Suppose $x \in \overline{Y}$. By local finiteness, there is some open neighborhood U of x that intersects only finitely many members of $\mathcal{A}$, say $A_1, \ldots, A_k$. For the sake of contradiction, suppose $x \notin \bigcup_{A \in \mathcal{A}} \overline{A}$. Then x is not in any of $\overline{A_1}, \ldots, \overline{A_k}$. Thus $V = U \setminus (\overline{A_1} \cup \cdots \cup \overline{A_k})$ is an open neighborhood of x. But V intersects no elements of $\mathcal{A}$, meaning it does not intersect Y, which contradicts $x \in \overline{Y}$. Thus $x \in \bigcup_{A \in \mathcal{A}} \overline{A}$, completing the proof.

[6.1.4] Definition (*Refinement, Countably Locally Finite*)

Let $\mathcal{A}$ be a covering of X. A collection $\mathcal{B}$ is a **refinement** of $\mathcal{A}$ if for each $B \in \mathcal{B}$, there is some $A \in \mathcal{A}$ such that $B \subseteq A$. If the elements of $\mathcal{B}$ are open, we call it an open refinement.

Furthermore, a collection $\mathcal{B}$ is **countably locally finite** (or σ-locally finite) if it can be written as a countable union of locally finite collections.

6.2 Paracompact Spaces

[6.2.1] Definition (*Paracompactness*)

A space X is said to be **paracompact** if every open covering $\mathscr{A}$ of X admits a locally finite open refinement $\mathscr{B}$ that covers X.

[6.2.2] Remark (*Intuition behind Paracompactness*)

Paracompactness is a natural generalization of compactness. Instead of demanding a finite subcover (which is often too strong of a global condition), we demand a locally finite open refinement. This ensures that while the total number of open sets in the cover might be infinite, locally around any point, the space only "sees" finitely many sets.

As we will see, this is precisely the topological machinery required to generalize partitions of unity. Because the refinement is locally finite, the seemingly infinite sum $\sum \varphi_k(x)$ evaluates to a finite sum in some neighborhood of any point x. Thus, the sum is automatically well-defined and continuous for free, without any need for analytic convergence tests.

[6.2.3] Theorem (*Paracompact Hausdorff Spaces are Normal*)

Every paracompact Hausdorff space is normal.

[6.2.4] Theorem (*Stone's Theorem*)

Every metrizable space is paracompact.

We omit the proofs of these theorems as they are highly technical, but their consequences are immense. Specifically, they allow us to generalize the existence of partitions of unity from compact spaces to any paracompact space.

[6.2.5] Theorem (*Existence of Partitions of Unity*)

Let X be a paracompact Hausdorff space, and let $\{U_\alpha\}_{\alpha\in J}$ be an indexed open covering of X. Then there exists a partition of unity $\{\varphi_\alpha\}_{\alpha\in J}$ dominated by $\{U_\alpha\}$.

This theorem is fundamentally why paracompactness is the standard assumption in differential geometry and functional analysis. It guarantees we can take locally defined objects (like coordinate charts, metric tensors, or differential forms) and stitch them together into global objects seamlessly, even if the manifold is not compact.

6.3 Metrization Theorems

The Urysohn metrization theorem gave us a sufficient condition for metrizability (regular and second-countable), but mathematicians are rarely satisfied until conditions are both necessary and sufficient. The following theorems, whose proofs we omit as they are largely technical extensions of Urysohn's methods, provide exact characterizations of metrizable spaces.

[6.3.1] Theorem (*Nagata-Smirnov Metrization Theorem*)

A space X is metrizable if and only if X is regular and admits a countably locally finite basis.

[6.3.2] Definition (*Locally Metrizable*)

A space X is **locally metrizable** if every point $x \in X$ has an open neighborhood U that is metrizable in the subspace topology.

[6.3.3] Theorem (*Smirnov Metrization Theorem*)

A space X is metrizable if and only if it is a paracompact Hausdorff space that is locally metrizable.

[6.3.4] Remark (*Manifolds are Metrizable*)

Smirnov's theorem is particularly elegant when applied to manifolds. Because an m-manifold is locally homeomorphic to $\mathbb{R}^m$, it is automatically locally metrizable. Thus, a manifold is globally metrizable if and only if it is paracompact Hausdorff. Since standard definitions of manifolds require them to be second-countable (which implies they are Lindelöf, and regular Lindelöf spaces are paracompact), manifolds are automatically metrizable!