4 Countability and Separation Axioms

4.1 The Countability Axioms

[4.1.1] Definition (*Countable Base, First Countable Space*)

Suppose $(X,\mathcal{T})$ is a topological space. We say some $x \in X$ admits a **countable basis** at x if there exists some countable collection $\mathcal{B}$ of open neighborhoods of x such that, for every open neighborhood U of x, U contains some element of $\mathcal{B}$. If all points in X satisfy this property, we say that X satisfies the **first-countability axiom**, or is **first-countable**.

[4.1.2] Example (*Metrizable Spaces are First-Countable*)

Suppose X is a metrizable topological space. For any $x \in X$, consider the collection of open neighborhoods $\mathcal{B} = \{B_n\}_{n\in\mathbb{Z}_+}$ about x such that $B_n = B(x, 1/n)$. Then, for any basic open neighborhood $U = B(x,\varepsilon)$ about x, we have that there is some $n \in \mathbb{Z}_+$ such that $1/n < \varepsilon$, and so $B_n(x, 1/n) \subseteq U$, making $\mathcal{B}$ a countable basis. Thus X is first-countable.

First countability is a property that lets convergent sequences probe limit points and the continuity of functions.

[4.1.3] Theorem

Let X be a topological space.

(1) Suppose $A \subseteq X$. If $(x_n) \subseteq A$ is a convergent sequence such that $x_n \to x$, then $x \in \overline{A}$. The converse is true if X is first-countable.

(2) If f is a continuous function, then for any $x_n \to x$, we have that $f(x_n) \to f(x)$. The converse is true if X is first-countable.

Proof.

(1) $(\Longrightarrow)$: This follows immediately by the definition of convergence of a sequence in a topological space.

$(\Longleftarrow)$: Suppose $x \in \overline{A}$. Note x admits some countable basis $\mathcal{B} = \{B_n\}_{n\in J}$. We can make a new countable basis from this, say $\mathcal{U} = \{U_n\}_{n\in J}$ such that $U_n = B_1 \cap \cdots \cap B_n$. Of course, each U_k is open and $x \in U_k$, so each U_k is an open neighborhood of x. Since $x \in \overline{A}$, every open neighborhood of x intersects A. Thus, for each $k \in J$, note that U_k is an open neighborhood of x, so we are justified in choosing x_k such that $x_k \in (U_k \cap A)$. If J is countably infinite, then $(x_n) \subseteq A$ is a sequence such that $x_n \to x$, obviously. If J is finite, let $n = \max(J)$. There exists some $k \in J$ such that, for any open neighborhood O of x, we have that $U_n \subseteq U_k \subseteq O$, with the first inclusion due to $\mathcal{U}$ defining a nested countable basis at x. Upgrade the list $(x_1, \ldots, x_n)$ to the sequence $(x_1, \ldots, x_n, x_n, x_n, \ldots)$, and note that $x_n \to x$.

(2) $(\Longrightarrow)$: Suppose f is continuous, and let $(x_n) \subseteq X$ be a sequence such that $x_n \to x$. Let V be some open neighborhood of $f(x)$, and by continuity of f, there exists an open neighborhood U of x such that $f(U) \subseteq V$. Note that there is some $N \in \mathbb{Z}_+$ such that $x_n \in U$ for every $n \geq N$, and since $f(U) \subseteq V$, we have that $f(x_n) \in V$ for every $n \geq N$, and so $f(x_n) \to f(x)$.

$(\Longleftarrow)$: We show f is continuous by showing that, for any $A \subseteq X$, we have that $f(\overline{A}) \subseteq \overline{f(A)}$. For any $x \in \overline{A}$, note that there is some sequence $(x_n) \subseteq A$ such that $x_n \to x$ by (1). By hypothesis, we have that $f(x_n) \to f(x)$, and we also have that $f(x) \in \overline{f(A)}$ from (1). Thus $f(\overline{A}) \subseteq \overline{f(A)}$, and so f is continuous.

❦

[4.1.4] Definition (*Second-Countability*)

If a topological space X admits a countable basis for its topology, then we say X satisfies the **second-countability axiom**, or is **second countable**.

Obviously, if X is second-countable, then it is first-countable, since every open neighborhood U of some $x \in X$ can be written as the union of a subset of a countable basis from X, and so x admits a countable basis.

[4.1.5] Example (*Examples of Second-Countable Spaces*)

Note that $\mathbb{R}$ admits a countable basis, where each element takes th eform (a, b) for rational endpoints $a, b \in \mathbb{Q}$. Similarly, $\mathbb{R}^n$ admits a countable basis where each element takes the form $\prod_{k=1}^{n}(a_k, b_k)$, where $a_k, b_k \in \mathbb{Q}$ for every k. Even $\mathbb{R}^\omega$ in its product topology admits a countable basis, with each set taking the form $\prod_{k=1}^{n}(a_k, b_k) \times \prod_{k \in \mathbb{Z}_+ \setminus \{1,\ldots,n\}} \mathbb{R}$ for $a_k, b_k \in \mathbb{Q}$ for each k.

The uniform topology on $\mathbb{R}^\omega$ is metrizable by the uniform metric $\overline{\rho}(x, y) = \sup_{n \in \mathbb{Z}_+}\{\overline{d}(x_n, y_n)\}$, where $\overline{d}(x_n, y_n) = \min(|x_n - y_n|, 1)$; hence, it is first-countable.

To show $\mathbb{R}^\omega$ is not second-countable, we appeal to the lemma that if a space X has a countable basis $\mathcal{B}$, then any discrete subspace $A \subseteq X$ is countable. Indeed, since each $\{a\}$ is open in A, there is some open set $U \subseteq X$ such that $U \cap A = \{a\}$. By the definition of a basis, there exists some $B_a \in \mathcal{B}$ with $a \in B_a \subseteq U$, yielding $B_a \cap A = \{a\}$. For any distinct $a, b \in A$, the basis elements B_a and B_b are not equal since $a \in B_a$ but $a \notin B_b$ (as $B_b \cap A = \{b\}$). The map $a \mapsto B_a$ is therefore an injection of A into $\mathcal{B}$, proving A must be countable.

Finally, consider the subspace $A \subseteq \mathbb{R}^\omega$ comprising all sequences of 0s and 1s. This set A is uncountable. However, for any distinct $a, b \in A$, they must differ in at least one coordinate, so $\overline{\rho}(a, b) = 1$. The subspace A therefore inherits the discrete metric, giving it the discrete topology. By contrapositive, $\mathbb{R}^\omega$ is not second-countable.

[4.1.6] Theorem (*First/Second-Countability Preserved Under Subspace and Product Topologies*)

A subspace of first/second-countable spaces is first/second-countable. The countable product of first/second-countable spaces is first/second-countable.

Proof. The proof is entirely mechanical and a straightforward application of the definition. ❦

[4.1.7] Definition (*Density*)

A set $A \subseteq X$ is said to be **dense** if $\overline{A} = X$.

[4.1.8] Definition (*Lindelöf Space*)

A space X is said to be **Lindelöf** if every open cover of X admits a countable subcover.

[4.1.9] Definition (*Separable Space*)

A space X is said to be **separable** if there exists a dense subset of X that is countable.

[4.1.10] Theorem

Suppose that X is second-countable.

(1) X is Lindelöf.

(2) X is separable.

That is, second-countability implies being Lindelöf and separable.

Proof. Let $\mathcal{B} = \{B_n\}$ be a countable basis for X.

(1) Let $\mathcal{A}$ be a cover for X. For each $x \in X$, there is some $A \in \mathcal{A}$ such that $x \in A$. Since A is open, there is a basic, open neighborhood B_n such that $x \in B_n \subseteq A$. Choose exactly one $A \in \mathcal{A}$ such that $B_n \subseteq A$, and let $A_n = A$. Then $\{A_n\}$ covers X, forming a countable subcover from $\mathcal{A}$.

(2) For each $n \in \mathbb{Z}_+$ such that $B_n \neq \emptyset$, choose some $x_n \in B_n$, and let D be a set comprising each x_n. Then note that for any basic open neighborhood B_k about any $x \in X$, $x_k \in B_k \cap D$, and so $\overline{D} = X$. Thus D is dense in X.

[4.1.11] Example (*Countability Properties of Sorgenfrey Line*)

Consider the set $\mathbb{R}_\ell$ endowed with the lower limit topology, also known as the *Sorgenfrey line*. We will show that it is first-countable, separable, and Lindelöf, but not second-countable.

Of course, any $x \in \mathbb{R}_\ell$ admits a countable basis $\{[x, x+1/n)\}_{n \in \mathbb{Z}_+}$, so $\mathbb{R}_\ell$ is first-countable. It is also easy to see that $\mathbb{Q}$ is dense in $\mathbb{R}_\ell$, and so $\mathbb{R}_\ell$ is separable.

We show $\mathbb{R}_\ell$ is not second-countable. Suppose $\mathcal{B}$ is a basis for the lower limit topology. Then, for any $x \in X$, let B_x be some basis element such that $x \in B_x \subseteq [x, x+1)$. For any $x \neq y \in \mathbb{R}_\ell$, note that $B_x \neq B_y$, since $[x = \min(B_x)] \neq [y = \min(B_y)]$. Thus the map from $\mathbb{R}_\ell$ to $\mathcal{B}$, defined by $x \mapsto B_x$, is injective, and so $|\mathbb{R}_\ell| < |\mathcal{B}|$, making the basis uncountable.

The proof of $\mathbb{R}_\ell$ being Lindelöf is annoying.

[4.1.12] Example (*Product of Lindelöf Spaces is not Lindelöf*)

Lindelöf-ness is not a property that is preserved under products. For example, even though $\mathbb{R}_\ell$ is Lindelöf, we can quickly show $\mathbb{R}_\ell \times \mathbb{R}_\ell$ (also known as $\mathbb{R}_\ell^2$, the *Sorgenfrey plane*) is not. Choose the basis that has sets of the form $[a, b) \times [c, d)$, and consider the set

$$L = \{(x, -x) : x \in \mathbb{R}_\ell\},$$

which is essentially the graph of $f(x) = -x$ in $\mathbb{R}_\ell^2$. It is quick to show L is closed, so we have that $\mathbb{R}_\ell^2 \setminus L$ is open. Then consider the open cover formed by $\mathbb{R}_\ell^2 \setminus L$ adjoined with open sets of the form $[a, b) \times [-a, c)$. Every basis element (except the first) intersects L, which is uncountable, and so there is no countable subcover. Thus $\mathbb{R}_\ell^2$ is not Lindelöf.

[4.1.13] Example (*Subspace of Lindelöf Space is not Lindelöf*)

The ordered square I_o^2 is compact (and thus Lindelöf), but the subspace $I \times (0, 1)$ is not Lindelöf.

[4.1.14] Recap

(1) A space X is **first-countable** at a point $x \in X$ (i.e., satisfies the **first-countability axiom** at x) if there exists a countable collection of open neighborhoods $\{B_n\}$ of x such that, for any open neighborhood U of x, there is some $n \in \mathbb{Z}_+$ for which $B_n \subseteq U$.

(2) From any countable basis $\{B_n\}$ at a point $x \in X$, one can derive a countable basis that is "nested" (i.e., $U_1 \supseteq U_2 \supseteq \cdots$). Indeed, let $U_n = B_1 \cap \cdots \cap B_n$. Then note x is in each B_k, and so x is in each U_k. Moreover, the finite intersection of open sets is open, so each U_k is an open neighborhood of x. Obviously $\{U_n\}$ forms a nested chain of open neighborhoods of x.

(3) First countability is a property that lets convergent sequences probe limit points and the continuity of functions. For a space X and some $A \subseteq X$, if we have a sequence $(x_n) \subseteq X$ such that $x_n \to x$, then $x \in \overline{A}$ by definition of convergence in a topological space. The converse is not necessarily true in a general topological space, but if X is first-countable, then it is immediate (simply take a countable basis at any $x \in X$, make it nested, then choose a sequence as one would in a metric space).

(4) Similarly, if $f : X \to Y$ is continuous, then for any $(x_n) \subseteq X$ such that $x_n \to x$, we have that $f(x_n) \to f(x)$ (simply by the property that if V is an open neighborhood of $f(x) \in f(X)$ for continuous f, there is some open neighborhood U of x such that $f(U) \subseteq V$). The converse is only true in a first-countable space, however. Using the previous theorem about sequences, one can prove that $f(\overline{A}) \subseteq \overline{f(A)}$.

(5) A space X is **second-countable** if it admits a countable basis.

(6) A subset $D \subseteq X$ is **dense** if $\overline{D} = X$.

(7) A space X is **Lindelöf** if every open covering of X admits a countable subcovering.

(8) A space X is **separable** if there exists a dense subset of X that is countable.

(9) Second-countability of a space X implies the space being Lindelöf and separable. If X is second-countable, it admits a countable basis $\{B_n\}_{n \in J}$. For any open covering $\mathcal{A}$ of X, note that for any $x \in X$, there is some $A \in \mathcal{A}$ such that $x \in A$. Since A is open, there is a basis element B_n such that $x \in B_n \subseteq A$. For all $x \in B_n$, choose this representative A again when querying the cover for a member that contains an element. Associating A with n by A_n, note that $\{A_n\}$ is an open subcover of X, making X Lindelöf. Furthermore, for each $n \in J$, choose some $x_n \in B_n$, and let D be the countable set of all x_n. For any $x \in X$, any basic open neighborhood intersects D (since it contains elements from every basic, open set), and so $\overline{D} = X$, making X separable as well.

4.1.1 Exercises

[4.1.15] Problem

For a topological space X, we say a subset $A \subseteq X$ is G_δ if A can be written as the countable intersection of open sets. Show that singletons are G_δ space in first-countable, T_1 topological spaces.

Proof. Let X be first-countable and T_1, and suppose $\{x\} \subseteq X$. Let $\{B_n\}_{n\in J}$ be a countable basis about x: we will show $\bigcap_{n\in J} B_n = \{x\}$. Simply, if $y \in \bigcap_{n\in J} B_n$, then y is in every open neighborhood of x, making y a limit point of $\{x\}$. But $\{x\}$ is closed in a T_1 space, menaing it contains all its limit points, and so y is forced to be x, completing the proof.

[4.1.16] Problem

Show every compact metrizable space X has a countable basis.

Proof. For any $n \in \mathbb{Z}_+$, note the set of all balls of radius $1/n$ forms an open cover for X, and by compactness we have a finite subcover $\mathcal{A}_n = \{A_n^i\}_{i=1}^k$. Then consider $\mathcal{B} = \bigcup_{n\in\mathbb{Z}_+} \mathcal{A}_n$, a countable set of open balls that cover X. To show $\mathcal{B}$ is a basis, we show that for any open neighborhood U of x, there is some $B \in \mathcal{B}$ such that $x \in B \subseteq U$.

Since U is open, there is some $\varepsilon > 0$ such that $B(x,\varepsilon) \subseteq U$. Choose some $n \in \mathbb{Z}_+$ such that $2/n < \varepsilon$. Since $\mathcal{A}_n$ covers X, there is some ball $B = B(p, 1/n) \in \mathcal{A}_n$ such that $x \in B$. Then note, for any $z \in B$, we have that

$$d(z,x) \le d(z,p) + d(p,x) < \frac{1}{n} + \frac{1}{n} = \frac{2}{n} < \varepsilon,$$

and so $x \in B \subseteq B(x,\varepsilon) \subseteq U$, completing the proof.

[4.1.17] Problem

Show that if a separable space is metrizable, then it is second-countable.

Proof. Suppose (X,d) has a countable subset $D = \{x_k\}_{k\in J}$, where J is countable, such that $\overline{D} = X$. For each $k \in J$, let $\mathcal{B}_k = \{B(x_k, 1/n)\}_{n\in\mathbb{Z}_+}$. Then note $\mathcal{B} = \bigcup_{k\in J} \mathcal{B}_k$ is a countable, open cover of X—we show that $\mathcal{B}$ forms a basis.

Let U be some open neighborhood of x, and let $\varepsilon > 0$ be such that $x \in B(x,\varepsilon) \subseteq U$. Since $\overline{D} = X$, any open neighborhood of x intersects some $x_k \in D$. Let $n \in \mathbb{Z}_+$ be such that $x_k \in B(x, 1/n) \cap D$, and also be such that $2/n < \varepsilon$. Thus $d(x, x_k) < 1/n$, and so for any $z \in B(x_k, 1/n)$, we have that

$$d(z,x) \le d(z,x_k) + d(x_k,x) < 2/n < \varepsilon,$$

and so $x \in \underbrace{B(x_k, 1/n)}_{\in\mathcal{B}} \subseteq B(x,\varepsilon) \subseteq U$.

[4.1.18] Problem

Show that if a Lindelöf space X is metrizable by a metric d, then it is second-countable.

Proof. For each $n \in \mathbb{Z}_+$, let $\mathcal{A}_n$ be the set of all open balls of radius $1/n$, which forms an open cover for X. Since X is Lindelöf, there is a countable subcover A_n. We conjecture that $\mathcal{B} = \bigcup_{n\in\mathbb{Z}_+} A_n$ is a countable basis for the metric topology of X. Of course, it is countable, and it covers X. We now show it has the basis property.

Let $x \in X$ be arbitrary, and let U be any open neighborhood of x. Since U is open, there is some $\varepsilon > 0$ such that $B(x, \varepsilon) \subseteq U$. Let $n \in \mathbb{Z}_+$ be such that $2/n < \varepsilon$. Note that $\mathcal{B}$ covers X with balls of radius $1/n$, and so there is some $B(y, 1/n) \in \mathcal{B}$ that is an open neighborhood for x. Then, for any $z \in B(y, 1/n)$, note that

$$d(z,x) \leq d(z,y) + d(y,x) < 2/n < \varepsilon,$$

and so $x \in \underbrace{B(y, 1/n)}_{\in\mathcal{B}} \subseteq B(x,\varepsilon) \subseteq U$. ❦

[4.1.19] Remark

We have shown that, in metrizable spaces, being Lindelöf, separable, and second-countable are all equivalent.

[4.1.20] Problem

Show that if X is separable, then every collection of disjoint, open sets in X is countable.

Proof. Let $D = \{x_n\}_{n\in J}$ be a countable subset of X such that $\overline{D} = X$ (i.e., D is dense in X). Let $\mathcal{A}$ be a collection of disjoint, open sets in X. Without loss of generality, suppose $\mathcal{A}$ does not contain the empty set. Since each $A \in \mathcal{A}$ is an open neighborhood of some element of X, there is some $k \in J$ for which $x_k \in A$ by the density of D in X. Since each $A \in \mathcal{A}$ is disjoint, the map from $\mathcal{A}$ to D given by $A \mapsto x_k$ is injective. Thus $|\mathcal{A}| \leq |D|$, making $\mathcal{A}$ countable. ❦

4.2 Separation Axioms

[4.2.1] Definition (*Topologically Distinguishability, Separation*)

Let x, y be two points in a topological space X. Then x, y are **topologically indistinguishable** if the set of all open neighborhoods about x coincides with the set of all open neighborhoods about y. Thus, x, y are **topologically distinguishable** if one point has an open neighborhood that excludes the other. Moreover, x, y are **separated** if there exist open neighborhoods about x and y that are disjoint. Separation generalizes easily to a point and a set or two sets.

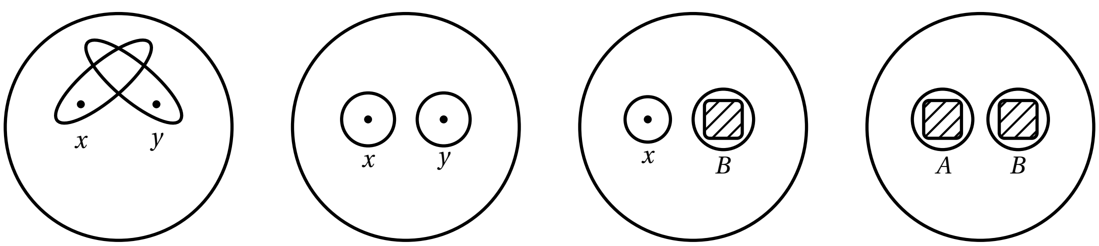
Note that when dealing with connected spaces, we also use separations, but only separations whose union is the entire space. In dealing with separation axioms, we still deal with separations (i.e., disjoint, open sets), but we do not care if their union is the entire space.

[4.2.2] Definition (*Regular, Normal Spaces*)

Suppose a space X is T_1 (i.e., all singletons are closed). We say that X is **regular** if, for any point x and a closed set $B \subseteq X$ disjoint from x, we may separate x and B. We say that X is **normal** if any pair of disjoint, closed sets can be separated.

Note that we specify B to be closed in the previous definition since, if $B \subseteq X$ is not closed, then we would be unable to separate a point $x \in \overline{B} \setminus B$ from B, making most spaces X not regular. The same is true for the definition of normality (consider a set A and a singleton set B that contains a point from $\overline{A} \setminus A$).

Obviously, a regular space is Hausdorff (consider a point x and the closed space $\{y\}$), and analogously, normal spaces are regular.



[4.2.3] Theorem (*Characterizations of Regularity and Normality*)

Suppose X is T_1.

(1) X is regular if and only if, for any $x \in X$ with any open neighborhood U of x, there is some open neighborhood V of x such that $\overline{V} \subseteq U$.

(2) **Shrinking Lemma.** X is normal if and only if, for any closed set $A \subseteq X$ contained in any open neighborhood U, there is some open set V that contains A such that $\overline{V} \subseteq U$.

Proof.

(1) ($\Longrightarrow$): Suppose X is regular. Consider a point $x \in X$ and some open neighborhood U of x. Set $B = X \setminus U$, a closed set; by regularity of X, there exist disjoint, open neighborhoods V, W of x and B. If we show that $\overline{V} \cap B = \emptyset$, then we have shown that $\overline{V} \subseteq U$. Indeed, if $y \in B$, then there is an open neighborhood W of y that does not intersect V, and so $y \notin \overline{V}$. Thus $\overline{V} \subseteq U$.

($\Longleftarrow$): Let $x \in X$ be a point disjoint from some closed set $B \subseteq X$. Then note $U = X \setminus B$ is an open neighborhood of x. By hypothesis, there is an open neighborhood V of x such that $\overline{V} \subseteq U$. Then $U, X \setminus \overline{V}$ are disjoint open neighborhoods separating x and B respectively.

(2) The proof is analogous to (1).

It is quick to see that the subspace V of a Hausdorff space X is Hausdorff. Indeed, if $x, y \in V$, then there exist disjoint open neighborhoods $X \cap V, Y \cap V$ of x, y respectively, where X, Y are disjoint open neighborhoods of x, y in the space X.

It is also quick to see that the product X of Hausdorff spaces $\{X_\lambda\}_{\lambda \in \Lambda}$ is Hausdorff. Indeed, for distinct $\mathbf{x} = (x_\lambda)_{\lambda \in \Lambda}$ and $\mathbf{y} = (y_\lambda)_{\lambda \in \Lambda}$, there is some $j \in \Lambda$ for which $x_j \neq y_j$. Since X_j is Hausdorff, there exist disjoint, open neighborhoods $O_x, O_y \subseteq X_j$ of x_j, y_j. Then the sets

$$O_X = O_x \times \left(\prod_{\lambda \in \Lambda \setminus \{j\}} X_\lambda \right) \quad O_Y = O_y \times \left(\prod_{\lambda \in \Lambda \setminus \{j\}} X_\lambda \right),$$

are disjoint, open neighborhoods of $\mathbf{x}$, $\mathbf{y}$ in the product topology.

[4.2.4] Theorem (*Subspace and Product of Regular Spaces is Regular*)

A subspace of regular spaces is regular. The product of a collection of regular spaces is regular.

Proof. Suppose X is a regular space and $Y \subseteq X$ is a subspace. Note Y is Hausdorff and thus T_1. Let x be a point in Y and B a closed subset of Y. Note that $\mathrm{Cl}_X(B) \cap Y = \mathrm{Cl}_Y(B) = B$, with the last equality by B closed. Considering $x \in X$ and $\overline{B} \subseteq X$ a closed set, by regularity of X we have that there exist disjoint, open neighborhoods U, V of x and $\overline{B}$. Then $U \cap Y, V \cap Y$ are disjoint, open neighborhoods of x and B, proving Y is regular.

Now suppose $\{X_\lambda\}_{\lambda \in \Lambda}$ is a collection of regular spaces. We show $X = \prod_{\lambda \in \Lambda} X_\lambda$ is regular by the alternative characterization. Note that X is Hausdorff and thus T_1, so it satisfies that part. Let $\mathbf{x} = (x_\lambda)_{\lambda \in \Lambda} \in X$, and let U be any open neighborhood of $\mathbf{x}$. Since U is open, there is a basic, open set $\prod_\lambda U_\lambda$ such that $x \in \prod_\lambda U_\lambda \subseteq U$. From this basis, we can easily construct an open neighborhood V of x such that $\overline{V} \subseteq U$. We will define $V = \prod_{\lambda \in \Lambda} V_\lambda$. Indeed, for each $\lambda \in \Lambda$, consider U_λ. If $U_\lambda = X_\lambda$, then let $V_\lambda = X_\lambda$. Otherwise, let V_λ be some open set such that $\overline{V_\lambda} \subseteq U_\lambda$. Then V is an open neighborhood for x, and since $\overline{V} = \prod_{\lambda \in \Lambda} \overline{V_\lambda}$, we have that $\overline{V} \subseteq \prod_{\lambda \in \Lambda} U_\lambda \subseteq U$, such that X is regular. ❦

[4.2.5] Example (*Hausdorff But Not Regular*)

Consider $\mathbb{R}_K$, the reals endowed with the topology given by basis elements of the form $(a, b) \setminus K$, where $K = \{1/n\}_{n \in \mathbb{Z}_+}$. It is quick to see $\mathbb{R}_K$ is Hausdorff: for any $x \neq y \in \mathbb{R}_K$, let O_x, O_y be disjoint, open neighborhoods of x, y in the standard topology for $\mathbb{R}$, then note $O_x \setminus K, O_y \setminus K$ are disjoint, open neighborhoods of x, y in the K-topology. However, it is not regular by a straightforward proof.

Note that normality of a space is not preserved under subspace nor product. For products, consider the Sorgenfrey line $\mathbb{R}_\ell$, which is normal, and the Sorgenfrey plane $\mathbb{R}_\ell^2$, which is not normal.

[4.2.6] Recap

(1) Two points x, y are **topologically indistinguishable** if the collection of open neighborhoods about x is the exact same as the collection of open neighborhoods about y (otherwise, the points are **topologically distinguishable**). We say two points x, y are **separated** if there exist disjoint, open neighborhoods of x and y. The same idea applies to separating a point from a subset or a subset from a subset.

(2) Suppose X is a T_1 space. Then X is **regular** if any point x disjoint from a closed set B can be separated. X is **normal** if any pair of disjoint, closed sets A, B can be separated.

(3) The requirement that the sets are closed are important. If we drop that restriction, then there is limit point disjoint from a non-closed set, but separating them is impossible, so regularity would become far too restrictive. The same idea applies for normal spaces.

(4) We may also characterize regularity and normality in a different way. A space X is regular if and only if, for any open neighborhood U of x, there is some open neighborhood V of x such that $\overline{V} \subseteq X$. Analogously, a space X is normal if and only if, for any open neighborhood U of a closed set A, there is an open neighborhood V of A such that $\overline{V} \subseteq A$.

(5) Note that the equivalent definition of regularity closely resembles local compactness. Recall that a space X is **locally compact Hausdorff** if, for any open neighborhood U of x, there is some open neighborhood V of x such that $\overline{V}$ is compact and $\overline{V} \subseteq X$. At once, we have that locally compact Hausdorff spaces are regular.

(6) Subspace and product operations on regular spaces produce regular spaces. This is not true for normal spaces.

4.2.1 Exercises

[4.2.7] Problem

Show that if X is regular, then prove that any two distinct points in X have open neighborhoods whose closures are disjoint.

Proof. Let $x \neq y \in X$. By Hausdorffness of X, we have disjoint, open neighborhoods U_x, U_y about x, y respectively. By regularity of X, there are neighborhoods V_x, V_y of x, y such that $\overline{V_x} \subseteq U_x$ and $\overline{V_y} \subseteq U_y$, and so $\overline{V_x} \cap \overline{V_y} = \emptyset$, completing the proof.

[4.2.8] Problem

Let $f, g : X \to Y$ be continuous maps into the Hausdorff space Y. Show the equalizer $\mathrm{Eq}(f, g) = \{x : f(x) = g(x)\}$ is closed in X.

Proof. We show $C = \mathrm{Eq}(f, g)$ contains all its limit points. For the sake of contradiction, suppose $x \in \overline{C} \setminus C$. By Hausdorffness of Y, there exist disjoint, open neighborhoods Y_f, Y_g of $f(x), g(x)$ respectively. Again by continuity of f, g, there exist open neighborhoods X_f, X_g of x for which $f(X_f) \subseteq Y_f$ and $g(X_g) \subseteq Y_g$. Then consider the single open neighborhood $U = X_f \cap X_g$ of x, which also satisfies $f(U) \subseteq Y_f$ and $g(U) \subseteq Y_g$. Since $x \in \overline{C}$, U intersects C at a point whose image under both f and g is y (since the point lies in the equalizer). Thus $y \in Y_f \cap Y_g$, a contradiction, and so the construction of x is invalid. Thus C contains all its limit points, completing the proof.

4.3 Normal Spaces

Normality isn't a *super* nice condition (recall that subspaces and products of normal spaces are not necessarily normal). However, many familiar classes of topological spaces are normal, and normality turns out to be an important condition for certain big theorems like Urysohn's lemma and Tietze's extension theorem.

[4.3.1] Theorem (*Regular, Second-Countable Spaces are Normal*)

A regular, second-countable space is normal.

Proof. Suppose X is a regular, second-countable space. Then X has a countable basis $\mathcal{B}$. Let A, B be any two closed subsets of X. Since A is closed, for any any $x \in A$, we may choose an open neighborhood U of x that is disjoint from B. By regularity, we know there is a neighborhood V of x for which $\overline{V} \subseteq U$. Since V is open, there is a basis element $O \in \mathcal{B}$ such that $x \in O \subseteq V$. Obviously, $\overline{O} \cap B = \emptyset$, since $\overline{O} \subseteq \overline{V}$. Identify each point x with some basis element as such, then let $\{U_n\}$ be the collection of all of these basis elements (note this is a subset of the countable basis, and so it is countable). Applying the same procedure to B, let the collection be $\{V_n\}$.

Now note $\bigcup U_n, \bigcup V_n$ are open sets containing A, B, but they are not necessarily disjoint. Instead, let us construct two open covers that *are* disjoint. For each n, define

$$U_n' = U_n \setminus \bigcup_{i=1}^{n} \overline{V_i} \quad V_n' = V_n \setminus \bigcup_{i=1}^{n} \overline{U_i}.$$

Note the finite union of compact spaces is compact, and as subspaces of a Hausdorff space, it is closed. Note each U_n', V_n' is a complement of a closed set with respect to the open sets U_n, V_n, so they are open. Moreover, for any $x \in A$, there is some $n \in \mathbb{Z}_+$ where $x \in U_n$ and $x \notin \overline{V_i}$ for each $i \in \{1, \ldots, n\}$ (obviously, by $\overline{V_i}$ being disjoint from A), and vice-versa holds for B.

We have shown $U' = \bigcup U_n'$ and $V' = \bigcup V_n'$ are open sets containing A and B, and now we show they are disjoint. For the sake of contradiction, suppose $x \in U' \cap V'$, and so $x \in U_j' \cap V_k'$ for some $j, k \in \mathbb{Z}_+$. If $j \le k$, then by definition of U_j', we have that $x \in U_j$; by definition of V_k', we have that $x \notin \overline{U_j}$, a contradiction.

[4.3.2] Theorem (*Metrizable Spaces Are Normal*)

A metrizable space is normal.

Proof. Let (X, d) be a metric space, and let $A, B \subseteq X$ be disjoint, closed sets. For each $a \in A$, there is some $\varepsilon_a > 0$ such that $B(a, \varepsilon_a) \cap B = \emptyset$, since otherwise a would be a limit point of B, forcing $a \in B$ by B being closed. An analogous procedure can be used to find ε_b for each $b \in B$. Finally, define

$$U = \bigcup_{a \in A} B(a, \varepsilon_a/2) \quad \text{and} \quad V = \bigcup_{b \in B} B(b, \varepsilon_b/2),$$

with $\varepsilon_a, \varepsilon_b$ as chosen before. Obviously, U and V form open sets that contain A and B. We prove they are disjoint by contradiction. Suppose $z \in U \cap V$, and so there exist $a \in A$ and $b \in B$ such that $z \in B(a, \varepsilon_a/2) \cap B(b, \varepsilon_b/2)$. Note that

$$d(a, b) \le d(a, z) + d(z, b) < (\varepsilon_a + \varepsilon_b)/2.$$

If $\varepsilon_a \le \varepsilon_b$, then $d(a, b) < \varepsilon_b$, and so $a \in B(b, \varepsilon_b/2)$. Vice-versa is true if $\varepsilon_b \ge \varepsilon_b$. But note the balls about points in B are disjoint from A by construction, and vice-versa for the balls constructed about points in A. Thus we have reached a contradiction, and so $U \cap V = \emptyset$, completing the proof.

[4.3.3] Theorem (*Compact Hausdorff Spaces are Normal*)

A compact Hausdorff space is normal.

Proof. Suppose X is compact Hausdorff. We first show X is regular. Let x be any point in X and B be any disjoint, closed set. For any $b \in B$, we have the existence of open neighborhoods U_b and V_b of x and b that are disjoint. Note that $\{V_b\}$ is an open cover for B from open sets of X, and since B is a closed subset of a compact space, there is a finite subcover $\{V_k\}_{k=1}^n$. Then $U = \bigcup_{b \in B} U_b$ and $V = \bigcap_{k=1}^n V_k$ are disjoint, open sets containing x and B respectively. (Alternatively, note X is compact Hausdorff and thus locally compact Hausdorff, then apply a previous theorem.)

For any two disjoint, closed sets A, B, produce disjoint, open neighborhoods for each $a \in A$ and B. Intersect a finite subcover of the open neighborhoods covering B, then note it is an open neighborhood disjoint from the open neighborhood about A, which is the union of each open neigbhorhood about every $a \in A$. ❦

[4.3.4] Theorem

All well-ordered sets (totally ordered sets with the least upper bound property) are normal in their order topology.

4.4 The Urysohn Lemma

The Ursyohn lemma states that for any normal space X and any pair of closed subsets $A, B \subseteq X$, there exists a continuous $f : X \to [a, b]$ (for some $a, b \in \mathbb{R}$) such that $f|_A \equiv a$ and $f|_B \equiv b$.

[4.4.1] Theorem (*The Urysohn Lemma*)

Let X be a normal space, and let A, B be disjoint, closed subsets of X. Then there is a continuous map

$$f : X \to [0, 1]$$

for which $f|_A \equiv 0$ and $f|_B \equiv 1$.

Before we prove this lemma, we consider a class of numbers known as **dyadic rationals**, a set of fractions whose denominator is a power of two. Namely, we consider the dyadic rationals contained in $[0, 1]$. Let D_n be the set of dyadic rationals of order n contained in $[0, 1]$ (i.e., pure fractions whose denominator is of the form 2^n). For example,

$$D_0 = \{0, 1\}, \quad D_1 = \left\{0, \frac{1}{2}, 1\right\}, \quad D_2 = \left\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\right\}, \quad \cdots .$$

Suppose we bisect the interval $[0, 1]$ n times. Then the endpoints of bisection correspond exactly with the values in D_n, motivating them geometrically. If we bisect the interval a countably infinite number of times (like we do in the proof of, say, Bolzano-Weierstrass), the endpoints yielded correspond exactly with the values in $D = \bigcup_{n \in \mathbb{Z}_+} D_n$. Crucially, D is dense in $[0, 1]$, so the open sets indexed by D are fine enough to bring about a function that satisfies the hypotheses.

We will bisect $[0, 1]$ a countably infinite number of times, use the shrinking lemma to force an open set each time we bisect, and index these open sets with the dyadic rationals. Then we will define a map using these open sets that satisfy the conditions of the hypothesis, completing the proof.

Proof. We start by creating our family of open sets indexed by D. We start with D_0: consider the set $U_1 = X \setminus B$. Then, since $A \cap B = 0$, we have that $A \subseteq U_1$. We have that A is closed and U_1 is open, and so by the shrinking lemma, there is an open set U_0 such that

$$A \subseteq U_0 \subseteq \overline{U_0} \subseteq U_1.$$

Now we move to D_1. Note that $\overline{U_0}$ is a closed subset of the open set U_1, so we apply the shrinking lemma again, yielding a set $U_{1/2}$ such that

$$\overline{U_0} \subseteq U_{1/2} \subseteq \overline{U_{1/2}} \subseteq U_1.$$

Now we move to D_2. Note that $\overline{U_0}$ is a closed subset of the open set $U_{1/2}$, and that $\overline{U_{1/2}}$ is a closed subset of the open set U_1. Then we have open sets $U_{1/4}, U_{3/4}$ such that

$$U_0 \subseteq \overline{U_0} \subseteq U_{1/4} \subseteq \overline{U_{1/4}} \subseteq U_{1/2} \subseteq \overline{U_{1/2}} \subseteq U_{3/4} \subseteq \overline{U_{3/4}} \subseteq U_1.$$

Repeat this process inductively, yielding the family of open sets $\{U_r\}_{r \in D}$. We are now ready to define the continuous function.

Define $f : X \to [0, 1]$ by the function

$$f(x) = \inf\{r \in D : x \in U_r\} \quad (\inf(\emptyset) = 1).$$

For any $x \in X$, we identify the smallest dyadic rational r for which the open set U_r contains x. If no such U_r exists, then f emits 1. Immediately, we note that $f|_A \equiv 0$, since $A \subseteq U_0$ and so every element of A is contained in U_0. Conversely, we note that $f|_B \equiv 1$, since $U_1 \subseteq B$ and so no element of B is contained in $\{U_r\}$.

Finally, we prove that f is continuous, completing the proof. We show continuity on the subbasis elements $[0, a)$ and $(a, 1]$ for any $a \in (0, 1)$. Note that if $f(x) < a$ if and only if $x \in U_r$ for dyadic rationals r such that $r < a$. Thus

$$f^{-1}([0, a)) = \bigcup_{r \in D, r < a} U_r.$$

Conversely, suppose $f(x) > a$. By density of D in $[0, 1]$, we can pick two dyadic rationals r, s such that $a < r < s < f(x)$. Since $s < f(x)$, we note that $x \notin U_s$. Since $\overline{U_r} \subseteq U_s$, note that $x \notin \overline{U_r}$ as well. Thus

$$f^{-1}((a, 1]) = \bigcup_{r \in D, r > a} (X \setminus \overline{U_r}).$$

Thus the preimage of all subbasis elements are open, meaning f is continuous. We have completed the proof. ❦

[4.4.2] Remark (*Urysohn's Lemma Applies to Any Closed Interval*)

Note the argument involved is invariant under translation and scaling of the codomain. Thus Urysohn's lemma guarantees any continuous map $f : X \to [a, b]$ for which $f|_A \equiv a$ and $f|_B \equiv b$.

[4.4.3] Definition (*Separated by a Continuous Function*)

Suppose X is a topological space and $A, B \subseteq X$ are subsets. If there exists a continuous function $f : X \to [0, 1]$ such that $f|_A \equiv 0$ and $f|_B \equiv 1$, then we say that A and B can be **separated by a continuous function**.

[4.4.4] Theorem (*Normal iff. Disjoint, Closed Sets Separable by Continuous Function*)

A space X is normal if and only if, any pair of disjoint, closed sets can be separated by a continuous function.

Proof. ($\Longrightarrow$): This is the statement of Urysohn's Lemma.

($\Longleftarrow$): Suppose any disjoint, closed subsets $A, B \subseteq X$ can be separated by a continuous function f. Then note $f^{-1}([0, 1/2])$ and $f^{-1}((-1/2, 1])$ are disjoint open neighborhoods for A and B. ❦

Note that the shrinking lemma applies to normal spaces, and so the proof for Urysohn's lemma cannot be lifted to regular spaces. That is, the statement that a point and a disjoint, closed set can be separated by a continuous function is *false*. Instead, we define a subclass of regular spaces for which this is true.

[4.4.5] Definition (*Completely Regular*)

A space X that satisfies the T_1 condition is called **completely regular** if any singleton subset $\{x_0\}$ and some disjoint, closed set B can be separated by a continuous function.

Immediately, we note that a completely regular space is regular: if a singleton subset is separated from a disjoint, closed set by a continuous function f, then $f^{-1}([0, 1/2))$ and $f^{-1}((1/2, 1])$ are the desired disjoint, open neighborhoods about the singleton and closed set. Furthermore, we note that a normal space is completely regular, as we simply take one closed set to be a singleton.

This class of spaces is particularly nice as well: the subspace and product operations leave its property intact.

[4.4.6] Theorem (*Subspace and Product of Completely Regular Space is Completely Regular*)

Suppose X is a completely regular space. Then any subspace $U \subseteq X$ is also completely regular. Moreover, suppose $\{X_\lambda\}_{\lambda \in \Lambda}$ is a collection of completely regular spaces. Then the product $X = \prod_{\lambda \in \Lambda} X_\lambda$ is also completely regular.

Proof. Note that U is regular—subspaces of regular spaces are regular—and thus T_1. Let $x_0 \in U$ be any point that is disjoint from the closed set $B \subseteq U$. Since B is closed in U, it contains all its limit points in U and so x_0 is not a limit point for B, meaning $x_0 \notin \mathrm{Cl}_X(B)$. Thus x_0 is disjoint from the closed set $\mathrm{Cl}_X(B)$, where both objects are treated as elements of X. By complete regularity, there is a separation by a continuous map $f : X \to [0, 1]$. Then $g = f|_U$ is a continuous map that separates x_0 and $\mathrm{Cl}_X(B) \cap U = \mathrm{Cl}_U(B) = B$, making U completely regular.

The proof for the product being completely regular, like the subspace, is extremely similar to the proof of products being regular, so we omit the proof. ❦

[4.4.7] Remark (*The Separation Axioms*)

The separation axioms have alternative names of the form T_j. They are organized such that a T_j space is automatically T_i, and that there exists one T_i space that isn't T_j (for $j > i$). They are as follows.

$\mathbf{T}_0$ (*Kolmogorov*). Any two points are topologically distinguishable.

$\mathbf{T}_1$ (*Accessible*). Any two points have open neighborhoods that exclude the other point.

$\mathbf{T}_2$ (*Hausdorff*). Any two points can be separated.

$\mathbf{T}_3$ (*Regular*). A T_1 space for which any point and disjoint, closed set can be separated.

$\mathbf{T}_{3.5}$ (*Completely Regular*). A T_1 space for which any point and disjoint, closed set can be separated by a continuous function.

$\mathbf{T}_4$ (*Normal*). A T_1 space for which any pair of disjoint, closed sets can be separated.

$\mathbf{T}_5$ (*Completely Normal*). A T_1 space that is normal and has the property that all its subspaces are normal.

4.5 Urysohn's Metrization Theorem

We apply Urysohn's lemma to prove a foundational metrization theorem, called Urysohn's metrization theorem. This theorem posits that regular, second-countable spaces are metrizable by embedding them into a metric space (namely, $[0,1]^\omega \subseteq \mathbb{R}^\omega$). There are two variations of this proof, one where $\mathbb{R}^\omega$ is endowed with the product topology, the other where it is endowed with the uniform topology: we take up the proof using the product topology.

[4.5.1] Theorem (*Urysohn's Metrization Theorem*)

Suppose X is a regular, second-countable space. Then X is metrizable.

Proof. We show X is metrizable by embedding it into a metrizable space Y. If f is the appropriate embedding, note that $(X \cong f(X)) \subseteq Y$. Since $f(X)$ is a subspace of the metric space Y, $f(X)$ takes on a metric topology, and by homeomorphism so does X. As aforementioned, we let Y be the space $\mathbb{R}^\omega$ together with its product topology, and we show there is an embedding of X into $[0,1]^\omega$.

We first prove that there exists a countable collection of continuous functions $f_n : X \to [0,1]$ such that, for any point $x_0 \in X$ and any neighborhood U of x_0, there exists some $n \in \mathbb{Z}_+$ such that f_n is positive at x_0 and vanishes outside of U. Since X is regular and second-countable, it is normal. By Urysohn's lemma, for any fixed $x_0 \in X$ and neighborhood U of x_0, the disjoint closed sets $\{x_0\}$ and $X \setminus U$ can be separated by a continuous function $f : X \to [0,1]$ such that $f(x_0) = 1$ and $f(X \setminus U) = \{0\}$. We now go one step further to show that we can extract a single *countable* collection of such functions that works for every possible (x_0, U) pair simultaneously.

Let $\{B_n\}_{n\in\mathbb{Z}_+}$ be a countable basis for X. For each pair of indices $m, n \in \mathbb{Z}_+$ for which $\overline{B}_n \subseteq B_m$, we apply Urysohn's lemma to separate the disjoint closed sets $\overline{B}_n$ and $X \setminus B_m$ by a continuous map $g_{m,n} : X \to [0,1]$ satisfying $g_{m,n}(\overline{B}_n) = \{1\}$ and $g_{m,n}(X \setminus B_m) = \{0\}$. Because this collection $\{g_{m,n}\}$ is indexed by a subset of $\mathbb{Z}_+ \times \mathbb{Z}_+$, it is countable and can be reindexed as $\{f_n\}_{n\in\mathbb{Z}_+}$.

To see that this collection satisfies the requirement, let $x_0 \in X$ and let U be an open neighborhood of x_0. By the definition of a basis, there is some B_m such that $x_0 \in B_m \subseteq U$. By regularity of X, there is some basis element B_n such that $x_0 \in B_n \subseteq \overline{B}_n \subseteq B_m$. The function $g_{m,n}$ is therefore defined for this pair; it yields $g_{m,n}(x_0) = 1 > 0$ and vanishes on $X \setminus B_m \supseteq X \setminus U$, completing the proof.

Now consider the map $F : X \to \mathbb{R}^\omega$ defined by $F(x) = (f_k(x))_{k\in\mathbb{Z}_+}$. At once we note that F is continuous since each component is continuous, a nice property of the product topology. Moreover, it is injective: for any $x \neq y$ in X, there exists an open neighborhood U of x that excludes y (by X being T_1). Then there is some $n \in \mathbb{Z}_+$ such that $f_n(x) > 0$ and $f_n(X \setminus U) = 0$ (and so $f(y) = 0$), meaning $F(x) \neq F(y)$. Thus F is a continuous bijection onto its image $Z = F(X)$, and our final task is to show F maps open sets to open sets.

Suppose $O \subseteq X$ is open: we seek to show $F(O)$ is open in $\mathbb{R}^\omega$. Since F is a bijection, there is some unique $x_0 \in X$ such that $F(x_0) = z_0$. Let $N \in \mathbb{Z}_+$ be the index for which $f_N(x_0) > 0$ and $f_n(X \setminus O) = \{0\}$. Finally, let

$$W = \underbrace{(\pi_N^{-1}((0,\infty)))}_{:=V} \cap Z.$$

Note that $(0,\infty)$ is open in $\mathbb{R}$, and by continuity of the projection map π_N, we have that V is an open subset of $\mathbb{R}^\omega$. Thus W is an open subset of Z in its subspace topology. We show that

$$z_0 \in W \subseteq F(O),$$

implying that $F(O)$ is open. Since $f_N(x_0) > 0$, we have that $z_0 \in [\pi_N^{-1}((0,\infty)) = V]$, and $z_0 \in Z$ is immediate by F being a bijection—thus $z_0 \in W$. Now let $w \in W$ be arbitrary. Since $Z \subseteq W$, there is some $x \in X$ such that $F(x) = z$, and so $\pi_N(z) \in (0,\infty)$. Then

$$\pi_N(z) = \pi_N(F(x)) = f_N(x) \in (0,\infty),$$

meaning $f_N(x) > 0$ and so $x \in O$. Thus $w \in F(O)$, meaning $W \subseteq F(O)$, and so $F(O)$ is open. Thus F is a homeomorphism onto an image (an embedding), completing the proof that X is metrizable. ❦

[4.5.2] Theorem (*Embedding Theorem*)

Let X be a T_1 space. Suppose that $\{f_\lambda\}_{\lambda\in\Lambda}$ is a collection of continuous maps $f : X \to \mathbb{R}$ satisfying the condition that, for any $x_0 \in X$ together with an open neighborhood U of x_0, there is some index $\lambda \in \Lambda$ such that $f_\lambda(x_0) > 0$ and $f_\lambda(X \setminus U) = \{0\}$. Then the function $F : X \to \mathbb{R}^\Lambda$, with codomain taking the product topology, defined by

$$F(x) = (f_\lambda(x))_{\lambda\in\Lambda}$$

is an embedding of X into $\mathbb{R}^\Lambda$.

The proof is drawn directly from the second part of the previous proof, simply replacing the indexing set $\mathbb{Z}_+$ with Λ. If each f_λ maps into $[0,1]$, as the case with the last proof, then we note F is an embedding of X into $[0,1]^\Lambda$, called the **Hilbert cube** in $\mathbb{R}^\Lambda$.

4.6 Tietze Extension Theorem

Another useful consequence of the Urysohn lemma is the useful theorem called the Tietze extension theorem. A common problem in analysis is being able to extend a construction to a larger space, while still keeping important properties intact. For a space X, we may have a continuous, real-valued function $f : A \to \mathbb{R}$, with A a subset of X. We ask if there is a way to extend f to some function $\tilde{f} : X \to \mathbb{R}$, such that $\tilde{f}|_A = f$ and that continuity is preserved. Of course, this is not true for all spaces X, but certain restrictions on X and A make this theorem true. This is what the Tietze extension theorem is concerned with.

[4.6.1] Theorem (*Tietze Extension Theorem*)

Suppose X is a normal space and $A \subseteq X$ is closed. Then any continuous map $f : A \to [a, b]$ may be extended to a continuous map $f : X \to [a, b]$. The result holds if the closed interval in the codomain is replaced with $\mathbb{R}$.

The proof will be omitted for now.

4.7 Embeddings of Manifolds

We have shown that regular, second-countable spaces can be embedded into $\mathbb{R}^\omega$. A natural question is to ask which type of space X can be embedded into some finite-dimensional Euclidean space $\mathbb{R}^n$. We show that compact manifolds satisfy this property.

[4.7.1] Definition (*m-Manifold*)

A space X is said to be an **m-manifold** if X is Hausdorff, second-countable, and is locally homeomorphic to $\mathbb{R}^m$. That is, for any $x \in X$, there is some open neighborhood U of x such that U is homeomorphic to some open subset of $\mathbb{R}^m$.

[4.7.2] Example (*Examples of Manifolds*)

A curve (with some regularity conditions imposed) is a 1-manifold, and the same for a surface being a 2-manifold.

[4.7.3] Definition (*Support of a Map*)

Suppose $\varphi : X \to \mathbb{R}$ is a real-valued map. Then the **support** of φ, denoted by $\operatorname{supp}(\varphi)$, is the closure of $\varphi^{-1}(\mathbb{R} \setminus \{0\})$.

[4.7.4] Remark (*Intuition behind Support*)

The support of a function is not simply where the points where the map is nonzero: it's the points where there are no open neighborhoods for which the map vanishes on. We can characterize the support of a function as where it's "active". A polynomial p may be zero at a root $x = r$, but it either rebounds or intersects without "wasting any time," and so we would say that the polynomial is "active" at r, and so $r \in \operatorname{supp}(p)$. The bottom line is, if a point x is not in $\operatorname{supp}(f)$, then there is some open neighborhood U of x such that $f|_U \equiv 0$. Thus the closure makes the support a much better tool to determine where a function is dead. We also note that the support being closed is a useful property (analogously, the complement of the support being open is useful).

[4.7.5] Definition (*Partition of Unity*)

Let $\{U_1, \ldots, U_n\}$ be some finite covering of the space X. Then a collection of continuous maps $\{\varphi_1, \ldots, \varphi_n\}$ such that $\varphi_k : X \to [0,1]$ is said to be a **partition of unity** dominated by $\{U_k\}$ if it satisfies the following conditions.

(1) $\operatorname{supp}(\varphi_k) \subseteq U_k$ for each $k \in \{1, \ldots, n\}$.

(2) For any $x \in X$, $\sum_{k=1}^n \varphi_k(x) = 1$.

[4.7.6] Remark (*Intuition behind Partition of Unity*)

Suppose X has a finite cover $\{U_1, \ldots, U_n\}$ together with a collection of continuous local maps $\{g_1, \ldots, g_n\}$ where each $g_k : U_k \to \mathbb{R}$. A natural question is: how can we produce a global map $g : X \to \mathbb{R}$ that represents these local maps best? There is no guarantee that g_j and g_k agree on the overlap $U_j \cap U_k$, so using the pasting lemma is not an option.

A naive approach would be to use characteristic functions χ_{U_k} to toggle the maps on and off, defining $g(x) = \sum_{k=1}^n \chi_{U_k}(x) g_k(x)$ (where the product is taken to be 0 outside U_k). However, this fails for two reasons. First, the binary nature of the characteristic function causes discontinuities at the boundaries. If a sequence of points inside U_k approaches a boundary point $p \in \partial U_k$, the term $\chi_{U_k}(x) g_k(x)$ approaches $g_k(p)$, which is generally nonzero. But for points immediately outside the boundary, the term is exactly 0. Unless the local map g_k happens to naturally vanish at the boundary of U_k, this creates a jump discontinuity. Second, in overlapping regions where multiple sets intersect, the values are blindly added together. If $x \in U_j \cap U_k$, the sum yields $g_j(x) + g_k(x)$, double-counting the outputs and artificially inflating the result.

A partition of unity solves this by acting as a continuous approximation of these characteristic functions. Rather than assigning a rigid binary 1 or 0 to dictate which open set x belongs to, $\varphi_k(x)$ assigns a “fractional ownership” to U_k. Deep inside U_k, away from other sets in the cover, $\varphi_k(x) = 1$ and acts exactly like a standard indicator function. In an overlap $U_j \cap U_k$, however, the functions provide a continuous interpolation that smoothly crossfades between the patches. For instance, suppose an overlap $V = U_j \cap U_k$ intersects no other sets U_i; by the first condition, all other maps φ_i must vanish identically on V. For any $v \in V$, the value $\varphi_j(v)$ measures how much “control” U_j exerts over v, and similarly for $\varphi_k(v)$. The second condition then forces $\varphi_j(v) + \varphi_k(v) = 1$, ensuring that the total ownership of the point is exactly 100%. Simply put, each map φ_k determines the amount of control U_k exerts over any point $x \in X$, with the restriction that U_k claims no ownership outside its boundary and that the the ownership each map exerts over x always sums perfectly to 1.

We define the global map as $g(x) = \sum_{k=1}^n \varphi_k(x) g_k(x)$, a convex combination of the local maps with the maps from the partition of unity. Since $\sum \varphi_k(x) = 1$, this behaves as a true weighted average. If the local maps g_j and g_k both output a value of c at an overlap, the global map outputs $0.7c + 0.3c = c$, which successfully blends conflicting local maps without inflating or deflating the values.

We will eventually prove compact manifolds may be embedded into Euclidean space by taking a finite cover of open sets, constructing local embedding maps from each open set to Euclidean space, then using a partition of unity to stitch together these maps and create an embedding of the manifold into Euclidean space.

[4.7.7] Theorem (*Existence of Partition of Unity*)

Let $\{U_1, \ldots, U_n\}$ be a finite open covering for the normal space X. Then X admits a partition of unity dominated by $\{U_k\}_{k=1}^n$.

Proof. We first show that any finite cover $\{U_k\}_{k=1}^n$ of X admits a finite cover $\{V_k\}_{k=1}^n$ such that $\overline{V_k} \subseteq U_k$ for each $k \in \{1, \ldots, n\}$. Define

$$A = U \setminus (U_2 \cup \cdots U_n).$$

Note that A is a closed subset of U_1, and by normality, there exists an open set V_1 such that $A \subseteq V_1 \subseteq \overline{V_1} \subseteq A$. Thus $\{V_1, U_2, \ldots, U_n\}$ is a finite open cover for U, since the missing elements from the incomplete cover $\{U_2, \ldots, U_n\}$ are in A, which is a subset of V_1. In general, if

$$\{V_1, \ldots, V_{k-1}, U_k, U_{k+1}, \ldots, U_n\}$$

covers X, let

$$A = X \setminus (V_1 \cup \cdots \cup V_{k-1} \cup U_{k+1} \cup \cdots \cup U_n).$$

Apply the same procedure to find V_k, then note $\{V_1, \ldots, V_k, U_{k+1}, \cdots, U_n\}$ covers X. Induction up to the n-th step shows this result is true.

Finally, let X be a normal space together with a finite open cover $\{U_1, \ldots, U_n\}$. By the previous theorem, choose $\{V_k\}_{k=1}^m$ and $\{W_k\}_{k=1}^m$ such that $\overline{V_k} \subseteq U_k$ and $\overline{W_k} \subseteq V_k$ for each $k \in \{1, \ldots, n\}$. Note that $\overline{W_k}$ and $X \setminus V_k$ are disjoint, closed sets in a normal space X, so Urysohn's lemma guarantees a continuous map $\psi_k : X \to [0,1]$ such that $\psi_k(\overline{W_k}) = \{1\}$ and $\psi_k(X \setminus V_k) = \{0\}$. Thus $\psi_k^{-1}(\mathbb{R} \setminus \{0\}) \subseteq V_k$, and so

$$\operatorname{supp}(\psi_k) \subseteq \overline{V_k} \subseteq U_k.$$

Define $\psi(x) = \sum_{k=1}^n \psi_k(x)$, which is strictly positive (find k such that $x \in U_k$, then note $\psi_k(x) > 0$). Then define

$$\varphi_k(x) = \frac{\psi_k(x)}{\psi(x)},$$

and note $\{\varphi_k\}_{k=1}^n$ is a partition of unity for X.

[4.7.8] Theorem

A compact manifold can be embedded into finite-dimensional Euclidean space.

Proof. Let X be a compact m-manifold. Note that by compactness and by m-manifolds being locally homeomorphic to $\mathbb{R}^m$, there exists a finite open cover $\{U_k\}_{k=1}^n$ comprised of open sets that are homeomorphic to open subsets of $\mathbb{R}^m$. Thus, for each $k \in \{1, \ldots, n\}$, there exists a map $g_k : U_k \to \mathbb{R}^m$ that is an embedding.

Since X is a compact manifold, it is compact Hausdorff and thus normal, so it admits a partition of unity $\{\varphi_k\}_{k=1}^n$ dominated by $\{U_k\}_{k=1}^n$. For each $k \in \{1, \ldots, n\}$, define $h_k : X \to \mathbb{R}^m$ by

$$h_k(x) = \begin{cases} \varphi_k(x) \cdot g_k(x) & x \in U_k, \\ \mathbf{0}_m & x \in X \setminus \operatorname{supp}(\varphi_k). \end{cases}$$

Note that on the intersection $U_k \cap (X \setminus \operatorname{supp}(\varphi_k))$, we have $\varphi_k(x) = 0$, so both branches agree with $\mathbf{0}_m$. Because U_k and $X \setminus \operatorname{supp}(\varphi_k)$ are open sets whose union is X, each h_k is well-defined and continuous by the pasting lemma.

Finally, set $N = n + nm$ and define the global map

$$g : X \to \mathbb{R}^N \cong \left(\underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{n \text{ times}} \times \underbrace{\mathbb{R}^m \times \cdots \times \mathbb{R}^m}_{n \text{ times}} \right)$$

by

$$g(x) = (\varphi_1(x), \ldots, \varphi_n(x), h_1(x), \ldots, h_n(x)).$$

We show that g is an embedding of X into $\mathbb{R}^N$. At once we have that g is continuous, since each of its component functions φ_k and h_k is continuous.

Next, we show that g is injective. Suppose $x, y \in X$ such that $g(x) = g(y)$. This implies $\varphi_k(x) = \varphi_k(y)$ and $h_k(x) = h_k(y)$ for every

$k \in \{1, \ldots, n\}$. Because $\{\varphi_k\}$ is a partition of unity, we have $\sum_{k=1}^{n} \varphi_k(x) = 1$, so there must exist at least one index k for which $\varphi_k(x) > 0$. Since $\varphi_k(y) = \varphi_k(x) > 0$, both points x and y lie in $\operatorname{supp}(\varphi_k) \subseteq U_k$. Evaluating h_k on U_k yields

$$\varphi_k(x) \cdot g_k(x) = h_k(x) = h_k(y) = \varphi_k(y) \cdot g_k(y).$$

Since $\varphi_k(x) = \varphi_k(y) > 0$, we may divide by this nonzero scalar to obtain $g_k(x) = g_k(y)$. Because g_k is an embedding on U_k, it is injective, forcing $x = y$. Thus, g is an injective continuous map.

Lastly, because X is compact and $\mathbb{R}^N$ is Hausdorff, the continuous injective map $g : X \to g(X)$ is a closed map, and therefore a homeomorphism onto its image. This completes the proof that g is an embedding into $\mathbb{R}^N$. ❦