3 Connectedness and Compactness

3.1 Connectedness

[3.1.1] Definition (*Separations and Connectedness*)

Let X be a topological space. A **separation** of X is a pair of nonempty, disjoint open subsets $U, V \subseteq X$ whose union is X. We say X is **connected** if there exists no separation of X.

[3.1.2] Remark (*Intuition Behind Separation*)

Intuitively, we could consider a topological space X to be disconnected if we can find two non-empty, disjoint subsets $A, B \subseteq X$ whose union is X, such that neither set contains a limit point of the other. The requirement that neither set possesses any of the other's limit points entails that A and B are topologically isolated from each other, incapable of touching even at their boundaries. This is the geometric idea behind a disconnection. If X cannot be partitioned into sets that strictly exclude each other's limit points, then no part of X can be cleanly detached from the rest, meaning the space is connected. The equivalence between this geometric idea and the framing of a disconnection in terms of open sets will be visited soon.

[3.1.3] Theorem

Separations are formed by clopen sets.

> **Proof.** Suppose X is a topological space, and suppose $A, B \subseteq X$ are disjoint, non-empty, open sets whose union is X. Note also that $B = X \setminus A$ and $A = X \setminus B$, which are closed sets, form a separation as well. Thus A, B are clopen. ❦

[3.1.4] Theorem

A topological space X is connected if and only if the only clopen subsets of X are $\emptyset$ and X.

> **Proof.**
>
> ($\Longrightarrow$): Suppose X is connected. Then let O be a clopen subset of X. Note that $O, X \setminus O$ are disjoint open subsets of X whose union is X. For X to be connected, it cannot be separated, meaning $O, X \setminus O$ cannot be non-empty. Thus the only clopen subsets of X are $\emptyset, X$.
>
> ($\Longleftarrow$): Suppose the only clopen subsets of X are $\emptyset, X$. By the previous theorem, there exists no separation of X, so X is connected. ❦

We reformulate the idea of connectedness in terms of limit points.

[3.1.5] Theorem

Let Y be a subspace of the topological space X. A separation of Y is defined to be a pair of disjoint, non-empty sets $A, B \subseteq Y$ such that neither contains the limit points of the other. Then Y is connected if no separation exists.

Proof. First, we show that disjoint, non-empty sets $A, B \subseteq Y$ such that neither contains the limit points of the other are open. Since $\mathrm{Cl}_Y(A) \cap B = A \cap \mathrm{Cl}_Y(B) = \emptyset$, and $\mathrm{Cl}_Y(O) = \mathrm{Cl}_X(O) \cap Y$ for a general $O \subseteq Y$, it follows that $\mathrm{Cl}_X(A) \cap B = A \cap \mathrm{Cl}_X(B) = \emptyset$. Thus $\mathrm{Cl}_X(A) \cap Y = A$ and $\mathrm{Cl}_Y(B) \cap Y = B$, meaning A, B are closed in Y. Thus A, B are open as well (since $B = A^c$ and $A = B^c$).

Suppose $A, B \subseteq Y$ are disjoint, non-empty open sets whose union is Y. Note that A, B are clopen, and so $A = \mathrm{Cl}_Y(A) = \mathrm{Cl}_X(A) \cap Y$, and the intersection of this with B is $\emptyset$. Analogously, $A \cap \mathrm{Cl}_Y(B) = \emptyset$ as desired. ❦

[3.1.6] Theorem

Suppose X is a topological space with some separation $C, D \subseteq X$. If a subspace $Y \subseteq X$ is connected, then Y lies entirely within either C or D.

Proof. Note separations are formed by open sets, so $C, D \subseteq X$ are open in X and consequently $C \cap Y, D \cap Y \subseteq Y$ are open. Then

$$(C \cap Y) \cap (C \cap D) = (C \cap D) \cap Y = \emptyset \cap Y = \emptyset,$$

$$(C \cap Y) \cup (C \cap D) = (C \cup D) \cap Y = X \cap Y = Y.$$

Y is connected and thus no separation for it exists, but $C \cap Y, D \cap Y$ would form a separation for Y if one is non-empty. Thus, $Y = C \cap Y$ or $Y = D \cap Y$, meaning Y is a subset of one of C, D. ❦

[3.1.7] Theorem (*Arbitrary Union of Connected Spaces are Connected*)

Suppose X is a topological space and $\{O_\lambda\}_{\lambda \in \Lambda}$ is a family of connected subsets of X, each containing some common point $p \in X$. Then $O = \bigcup_{\lambda \in \Lambda} O_\lambda$ is connected.

Proof. Suppose X is a topological space and $\{O_\lambda\}_{\lambda \in \Lambda}$ is a family of connected subsets of X, each containing some common point $p \in X$. Let $O = \bigcup_{\lambda \in \Lambda} O_\lambda$. For the sake of contradiction, suppose $A, B \subseteq O$ is a separation for O. Without loss of generality, suppose $p \in A$. Then note each O_λ is connected, and so $O_\lambda \subseteq A$ is forced ($O_\lambda \not\subseteq B$ since $p \notin B$). Thus $O \subseteq A$, meaning $B = \emptyset$, a contradiction. ❦

[3.1.8] Theorem (*Continuous Image of Connected Set is Connected*)

The continuous image of a connected set is connected.

Proof. Let $f : X \to Y$ be a continous map from a connected topological space X to some set Y. Note then that $\tilde{f} : X \to f(X)$ is continuous—we seek to show $f(X)$ is connected. For the sake of contradiction, suppose $A, B \subseteq f(X)$ forms a separation for $f(X)$. Note that

$$X = \tilde{f}^{-1}(f(X)) = \tilde{f}^{-1}(A \cup B) = \tilde{f}^{-1}(A) \cup \tilde{f}^{-1}(B),$$

where the last expression is the union of two open sets by continuity of $\tilde{f}$. Then this is a separation of X, but X is connected, which is a contradiction. ❦

An immediate corollary to this theorem is the Intermediate Value Theorem, provided we have proven connected subsets of $\mathbb{R}$ are equivalent to intervals. We prove this later.

[3.1.9] Theorem (*Finite Products of Connected Sets are Connected*)

A finite product of connected topological spaces is connected.

Proof. Let X, Y be connected topological spaces: we seek to show $X \times Y$ is connected as well. To start, fix $(a, b) \in X \times Y$. Then note

$$X \cong (X \times b) \qquad Y \cong (a \times Y).$$

Since connectedness is a topological property, it is preserved under homeomorphism. Thus

$$T_{(a,b)} = (X \times b) \cup (a \times Y)$$

is a union of two connected subspaces of $X \times Y$ that contains the point (a, b), so it is also connected. Then note

$$X \times Y = \bigcup_{(a,b) \in X \times Y} T_{(a,b)},$$

and so $X \times Y$ is connected. We may generalize this to finitely many products by induction. ❦

We will prove later that $\mathbb{R}$ is connected, but it is pretty intuitive to understand why. We discuss another advantage of the product topology over the box topology, in the fact that the product topology preserves more topological properties than the box topology.

[3.1.10] Remark (*Connectedness of $\mathbb{R}^\omega$ Under Box and Product Topologies*)

Consider $\mathbb{R}^\omega$ endowed with the box topology. Then note the set of bounded sequences, say O, and unbounded sequences, say O^c, form a partition for $\mathbb{R}^\omega$. We simply show these sets are open in the box topology. For any $\mathbf{a} \in \mathbb{R}^\omega$, define

$$U_a = (a_1 - 1, a_1 + 1) \times (a_2 - 1, a_2 + 1) \times \cdots,$$

which is open in the box topology. Then note each $U_\mathbf{a} \subseteq O$ for a bounded sequence $\mathbf{a}$, and so $O = \bigcup_{\mathbf{a} \in O} U_\mathbf{a}$, and so O is open. O^c can be proven to be open similarly, so O, O^c form a separation for $\mathbb{R}^\omega$ under the box topology.

Now consider $\mathbb{R}^\omega$ endowed with the product topology. For each $n \in \mathbb{Z}_+$, define $\tilde{\mathbb{R}}^n$ as the subspace of $\mathbb{R}^\omega$ that is the set of all sequences $\mathbf{x} = (x_1, x_2, \dots)$, where each $x_i = 0$ for every $i > n$. Of course, $\tilde{\mathbb{R}}^n \cong \mathbb{R}^n$ for every $n \in \mathbb{Z}_+$, so these subspaces are connected. We define $\mathbb{R}^\infty = \bigcup_{n \in \mathbb{Z}_+} \tilde{\mathbb{R}}^n$, which is connected since each space share the common zero sequence. From an intuitive standpoint, we recognize $\mathbb{R}^\infty$ as the set of all sequences whose tails zero out. We seek to show that $\mathbb{R}^\omega$, endowed with the product topology, is the topological closure for $\mathbb{R}^\infty$—since the closure of a connected set is connected, our work will be done. For any $\mathbf{a} \in \mathbb{R}^\omega$, we show that any open neighborhood about $\mathbf{a}$ intersects $\mathbb{R}^\infty$. Let O be a basic open set of $\mathbb{R}^\omega$ that contains $\mathbf{a}$. In the product topology, this means there is some $n \in \mathbb{Z}_+$ such that

$$O = \left(\prod_{k=1}^{n} (a_k - \varepsilon_k, a_k + \varepsilon_k) \right) \times \prod_{n \in \mathbb{Z}_+ \setminus \{1, \cdots, n\}} \mathbb{R}.$$

Thus $(a_1, a_2, \dots, a_n, 0, \dots) \in O$, so $O \cap \mathbb{R}^\infty \neq \emptyset$.

Even though $\mathbb{R}$ is connected, the box topology does not preserve connectedness when $\mathbb{R}$ is multiplied by itself countably many times, whereas the product topology does. In fact, we prove later that an arbitrary product of connected sets is connected under the product topology.

3.1.1 Exercises

[3.1.11] Problem

Let X be a topological space, and let $\{A_n\}_{n \in \mathbb{Z}_+}$ be a family of connected topological subsets of X such that $A_n \cap A_{n+1} \neq \emptyset$. Show $A = \bigcup_{n \in \mathbb{Z}_+} A_n$ is a connected subset of X.

> **Proof.** For the sake of contradiction, suppose $A = B \cup C$ is a separation. Recall that each A_n is a connected subset of X: if A_n were disconnected in A, then a separation of it would exist in A (and thus consequently X), so A_n is also a connected subset of A in its subspace topology. Without loss of generality, we can say $A_1 \subseteq B$. Then $A_2 \subseteq B$ or $A_2 \subseteq C$, but $A_1 \cap A_2 \neq \emptyset$ and so $A_2 \subseteq B$ is forced. By induction, it follows every $A_n \subseteq B$, and so $A \subseteq B$ and $C = \emptyset$, a contradiction. ❦

[3.1.12] Problem

Let $A \subseteq X$. Suppose $C \subseteq X$ is a connected subspace of X that intersects A and $X \setminus A$. Show that $C \cap \partial A \neq \emptyset$.

> **Proof.** Note that since $C \cap A \neq \emptyset$ and $C \cap A^c \neq \emptyset$, it follows that $C \cap \overline{A} \neq \emptyset$ and $C \cap \overline{(X \setminus A)} \neq \emptyset$. Moreover, note
>
> $$C = C \cap X = C \cap (\overline{A} \cup \overline{(X \setminus A)}) = \underbrace{(C \cap \overline{A})}_{:=A} \cup \underbrace{(C \cap \overline{(X \setminus A)})}_{:=B}.$$
>
> If $A \cap B = \emptyset$, then each set contains neither of the other's limit points and so this would be a separation of C, which is impossible since C is connected. Thus there is some $p \in A \cap B$, implying $p \in C \cap \partial A$. ❦

[3.1.13] Problem

Let $\{X_\lambda\}_{\lambda\in\Lambda}$ be a family of connected spaces, and let $X = \prod_{\lambda\in\Lambda} X_\lambda$. Fix $a = (a_\lambda)_{\lambda\in\Lambda}$.

(1) For any finite $K \subseteq \Lambda$, define X_K to be the set of all points $x = (x_\lambda)_{\lambda\in\Lambda}$ such that $x_\lambda = a_\lambda$ for every $\lambda \notin K$. Show X_K is connected.

(2) Show that $Y = \bigcup_{K\subseteq\Lambda, |K|<\infty} X_K$ is connected.

(3) Show that $X = \overline{Y}$ and conclude that X is connected.

This result shows that the arbitrary product of connected sets is connected as well.

Proof.

(1) Note that
$$X_K \cong \prod_{\lambda\in K} X_\lambda \times \prod_{\lambda\notin K} \{a_\lambda\} = C \times \prod_{\lambda\notin K} \{a_\lambda\},$$
where C is the product of finitely many connected sets. Moreover, note $\prod_{\lambda\notin K}\{a_\lambda\}$ is a singleton, which is trivially connected. Thus, the product of these two yields a connected subset of X, and since it is homeomorphic to X_K, we are done.

(2) Note each X_K, for finite K, is connected. Moreover, each shares the common point a, so their union is connected.

(3) Note that $Y \subseteq X$, so $\overline{Y} \subseteq X$ immediately. Thus we show every $X \subseteq \overline{Y}$. Let $x \in X$, and let O be a basic open neigbhorhood of x in the product topology of X. Then note $O = \prod_{\lambda\in\Lambda} U_\lambda$, where $U_\lambda = X_\lambda$ for all but finitely many valued of λ, which we say are $\lambda_1, \ldots, \lambda_n$. Thus there is some point in O with arbitrary terms for indices $\lambda_1, \ldots, \lambda_n$ (such that they satisfy the product topology condition) and a_λ for all other values of λ. This point also belongs to Y, so $O \cap Y \neq \emptyset$. Since this basic open set is arbitrary, it follows that $X \subseteq \overline{Y}$, completing the proof.

3.2 Connected Subspaces of the Real Line

[3.2.1] Definition (*Linear Continuum*)

A totally ordered set X with more than one element is said to be a **linear continuum** if the following hold:

(1) Any nonempty subset of X that is bounded above has a supremum.

(2) If $x < y \in X$, there is some $z \in X$ such that $x < z < y$.

[3.2.2] Theorem (*Connectedness of Linear Continuums*)

If L is a linear continuum in the order topology, then L is connected, and so are intervals/rays in L.

Proof. Recall that a subset $Y \subseteq L$ is said to be **convex** if, for any $a < b \in Y$, the interval $[a, b]$ of Y is a subset of Y (that is, all elements of L in between a and b lie in L). Note L is a convex subset of itself, and the intervals and rays in L are exactly the convex subsets of L, so let Y be an arbitrary convex subset of L—we prove it is connected.

For the sake of contradiction, suppose $Y = A \cup B$ is a separation. Pick some $a \in A$ and $b \in B$ (without loss of generality, let $a < b$), then note $[a, b] \subseteq Y$ by convexity of Y. Since A, B form a cover for Y, note that

$$A_0 = A \cap [a, b] \quad B_0 = B \cap [a, b]$$

forms a cover for $[a, b]$. Trivially, it follows that A_0 and B_0 are disjoint and open in the subspace topology $[a, b] \subseteq Y$. Thus $[a, b] = A_0 \cup B_0$ is a separation. Now let $c = \sup(A_0)$, which is well-defined since A_0 is a nonempty, bounded subset of a linear continuum. Since $[a, b]$ contains its maximum element b, it follows that $c \leq b$ and so $c \in [a, b]$. We show $c \notin A_0 \cup B_0 = [a, b]$ to yield a contradiction.

First, suppose $c \in B_0$. Then $c \neq a$, since $A_0 \cap B_0 = \emptyset$. By virtue of B_0 being open in $[a, b]$, there is some $a < d < b$ such that $(d, c] \subseteq B_0$. Then note we have have $(d, b] = (d, c] \cup (c, b]$. Indeed, $(c, b] \cap A_0 = \emptyset$ since c is greater than every element in A_0. Moreover, $(d, c] \subseteq B_0$, and so $(d, c] \cap A_0 = \emptyset$ since $B_0 \cap A_0 = \emptyset$. Thus d is an upper bound for A that is less than c, a contradiction.

Now suppose $c \in A_0$. Then note $c \neq b \in B_0$, so $a \leq c < b$. Since $A_0 \subseteq [a, b]$ is open, there is some $e \in A_0$ such that $[c, e) \in A_0$. Then there is some $c < z < e$ such that $z \in A_0$, a contradiction since c is meant to be an upper bound.

Thus Y cannot be separated. ❦

Since $\mathbb{R}$ is a linear continuum, it follows that $\mathbb{R}$ and all intervals/rays of $\mathbb{R}$ are connected. An immediate corollary is the Intermediate Value Theorem.

[3.2.3] Theorem (*Intermediate Value Theorem*)

Suppose $f : X \to Y$ is a continuous map from a connected set X to some ordered set Y in its order topology. For any two points $a, b \in X$, and for any $r \in Y$ such that $f(a) < r < f(b)$, then there is some $c \in X$ such that $f(c) = r$.

Proof. Let $A = f(X) \cap (-\infty, r)$ and $B = f(X) \cap (r, \infty)$. Note that A, B are disjoint and non-empty, since $f(a) \in A$ and $f(b) \in B$. Each is open in $f(X)$, being the intersection of an open set in Y (a ray) with $f(X)$ itself. Thus, if there is no $c \in X$ such that $f(c) = r$, then $f(X) = A \cup B$ would constitute a separation, which is a contradiction since the continuous image of a connected set is connected. Thus such a c exists. ❦

[3.2.4] Definition *(path-connectedness)*

Given points $x, y \in X$, we say a **path** from x to y is some continuous function $f : [a, b] \to X$ (where $[a, b] \subseteq \mathbb{R}$) such that $f(a) = x$ and $f(b) = y$. A space X is said to be **path-connected** if every two points in X can be joined by a path.

[3.2.5] Theorem *(path-connectedness Implies Connectedness)*

Suppose X is path-connected. Then X is connected.

Proof. Suppose X is path-connected, and for the sake of contradiction, let $X = A \cup B$ constitute a separation. For some $x \in A$ and $y \in B$, let $f : [a, b] \to X$ be a path from x to y. Then $f([a, b])$ is a connected subset of X, so it belong entirely in either A or B, a contradiction since its image has elements in both sets. ❦

3.2.1 Exercises

[3.2.6] Problem

(1) Show no two of $(0, 1), (0, 1], [0, 1]$ are homeomorphic.

(2) Suppose there exist embeddings $f : X \to Y$ and $g : Y \to X$. Give an example of X, Y that are not homeomorphic.

Proof.

(1) Suppose $f : (0, 1) \to (0, 1]$ is a homeomorphism. Since f is surjective, there is some $c \in (0, 1)$ such that $f(c) = 1$. Then define $\tilde{f}$ by a domain restriction $\tilde{f} : (0, 1) \setminus \{c\} \to (0, 1)$, which is also a homeomorphism. But then note that even though the domain of $\tilde{f}^{-1}$ is connected, its image is not, a contradiction. An analogous argument works for all other pairs.

(2) Consider $X = (0, 1]$ and $Y = [0, 1]$. Then let f be the identity and let $g(x) = 0.5x + 0.5$.

❦

[3.2.7] Problem

Suppose X is an ordered set in the order topology. Show that if X is connected, then X is a linear continuum.

Proof. Fix $a < b \in X$. If there were no $c \in X$ such that $a < c < b$, then b is the immediate successor to a, and so $X = (-\infty, b) \cup (a, \infty)$ constitutes a separation for X, which is impossible—thus $c \in X$.

Now suppose $O \subseteq X$ is nonempty and bounded above—we show there exists a supremum for O. Let A be the set of all upper bounds on O, and for the sake of contradiction, suppose A has no least element. Thus, for any $a \in A$, there is some $a' \in A$ such that $a' < a$. Note then that A contains no limit points of A^c and vice versa, so $A \cup A^c = X$ is a separation, a contradiction. ❦

[3.2.8] Problem

Let X, Y be ordered sets in the order topology. Show if $f: X \to Y$ is order-preserving and surjective, then f is a homeomorphism.

Proof. Immediately, f is injective, since if $a \neq b$, it follows that $f(a) \neq f(b)$ by hypothesis—thus f is a bijection. Let $B = (a, b)$ be a basic open set in X (that is, either an interval or ray). For any $x < y \in B$, it follows that $f(x) < f(y)$ by hypothesis. If there is some $c \in Y$ such that $f(x) < c < f(y)$, then by surjectivity there is some $r \in X$ such that $f(r) = c$. By order preserving properties of f, r must be such that $x < r < y$, and so $r \in B$. Thus f sends an interval/ray in X to an interval/ray in Y. Note that $x \in B$ if and only if $a < x < b$; coupling this with the convexity of $f(B)$, we have that $f(B) = (f(a), f(b))$, an open interval in Y. An analogous argument can be used for f^{-1} as well, which proves f is a homeomorphism. ❦

[3.2.9] Problem

(1) Is a product of path-connected spaces necessarily path-connected?

(2) Suppose $A \subseteq X$ is such that A is path-connected. Is $\overline{A}$ path-connected?

(3) Suppose X is path-connected and $f: X \to Y$ is a continuous map. Is $f(X)$ path-connected?

(4) Suppose $\{X_\lambda\}_{\lambda \in \Lambda}$ is a set of path-connected subsapces of X. If $\bigcap_{\lambda \in \Lambda} X_\lambda \neq \emptyset$, is it true that $\bigcup_{\lambda \in \Lambda} X_\lambda$ is path-connected?

Proof.

(1) Let $\{X_\lambda\}_{\lambda \in \Lambda}$ be a set of path-connected topological spaces, and let $X = \prod_{\lambda \in \Lambda} X_\lambda$. Let $\mathbf{x} = (x_\lambda)_{\lambda \in \Lambda}, \mathbf{y} = (y_\lambda)_{\lambda \in \Lambda} \in X$ be points. Note that for each λ, there is a continuous path $\gamma_\lambda : [a, b] \to X_\lambda$ such that $\gamma_\lambda(a) = x_\lambda$ and $\gamma_\lambda(b) = y_\lambda$. Now we may define

$$\gamma : [a, b] \to X \qquad \gamma(t) = (\gamma_\lambda(t))_{\lambda \in \Lambda},$$

then we see $\gamma(a) = \mathbf{x}$ and $\gamma(b) = \mathbf{y}$. Moreover, a map into the product topology is continuous so long as the components are continuous. Thus the product of path-connected spaces is path-connected.

(2) No. Let A be the image of the topologist's sine curve.

(3) Let $x, y \in X$. Then $x, y \in f^{-1}(f(X)) \subseteq X$. By path-connectedness of X, there is some $\gamma : [a, b] \to f^{-1}(f(X))$ such that $\gamma(a) = x$ and $\gamma(b) = y$. Then consider $f \circ \gamma : [a, b] \to f(X)$. $f \circ \gamma$ is the composition of continuous functions, so it is continuous. Moreover, $(f \circ \gamma)(a) = f(x)$ and $(f \circ \gamma)(b) = f(y)$. Thus this is a continuous path from $f(x)$ to $f(y)$, and since these are arbitrary points of $f(X)$, we conclude $f(X)$ is path-connected.

(4) Suppose $x, y \in \bigcup_{\lambda \in \Lambda} X_\lambda$: more specifically, let λ_x and λ_y be such that $x \in X_{\lambda_x}$ and $y \in X_{\lambda_y}$. By hypothesis there is some $z \in \bigcap_{\lambda \in \Lambda} X_\lambda$, so z is in both subspaces. By path-connectedness of each space, note there exists paths from x to z and y to z in the respective spaces. Merging these paths gives a new path from x to y that is contained in the union of all the path-connected spaces, completing the proof.

❦

[3.2.10] Problem

Suppose $U \subseteq \mathbb{R}^2$ is an open, connected subspace of $\mathbb{R}^2$. Show U is path-connected.

Proof. Fix $x_0 \in U$, and let $O \subseteq U$ be the set of all points $x \in U$ such that there is a path from x to x_0. Let $x \in O$ be arbitrary: since $x \in U$ as well, there is some open ball B such that $x \in B \subseteq U$. For every $y \in B$, note there is a path from y to x (since balls in $\mathbb{R}^2$ are path-connected), and there is a path from x to x_0, so there is a path from y to x_0. Thus $y \in O$, and since y is arbitrary, this implies $B \subseteq O$. Since there is a open set about each point of O contained in O, we have that O is open.

Now consider O^c. Once again, let $x \in O^c$ be arbitrary, then there is some open ball B such that $x \in B \subseteq U$. For the sake of contradiction, suppose there is some $y \in B \cap O$. Then there is a path connection $x_0 \to y \to x$, which is a contradiction, since $x \in O$ in this case. Thus $B \subseteq O^c$ and so O^c is open.

Finally, note that O^c are disjoint, open sets such that $U = O \cup O^c$. Since U is connected, no separation exists, so one of these sets must be empty. Certainly, $x_0 \in O$, so O^c is empty and thus $O = U$, meaning U is path-connected. ❦

3.3 Components and Local Connectedness

This section seeks to formalize the idea that, for any topological space X, there is a cover of connected and path-connected subsets of X.

[3.3.1] Definition (*Connected Components*)

Suppose X is a topological space. Define $\sim$ to be an equivalence relation on X such that, if $x \sim y$, then there is some connected subspace $C \subseteq X$ such that $x, y \in C$. The equivalence classes for $\sim$ are called the **connected components** of X.

Proof.

(1) **Reflexivity.** $x \sim x$ since $\{x\} \subseteq X$ is a connected subspace.

(2) **Symmetry.** Immediate.

(3) **Transitivity.** If $x \sim y$ and $y \sim z$, then there are connected subsets $A, B \subseteq X$ where $x, y \in A$ and $y, z \in B$. Then $A \cup B$ is a union that shares the point y, so it is connected, and it contains x, z. Thus $x \sim z$.

❦

[3.3.2] Theorem

The connected components of X are connected, disjoint subspaces of X whose union is X. Every nonempty connected subspace of X is fully contained in one equivalence class (i.e., it intersects exactly one equivalence class).

Proof. Since the connected components of X are equivalence classes, they form a partition for X. If $C \subseteq X$ were connected and were such that $C \cap [x_1]_\sim \neq \emptyset$ and $C \cap [x_2]_\sim \neq \emptyset$, where $\sim$ is the equivalence relation that induces the components, then $x_1 \sim x_2$ and so $[x_1]_\sim = [x_2]_\sim$—thus C intersects exactly one equivalence class and is also fully contained in one.

Fix some $x_0 \in X$, and note $[x_0]_\sim$ is a component. For any $x \in [x_0]_\sim$, there is some connected subspace C_x for which $x, x_0 \in C_x$. With the previous part in mind, note $C_x \subseteq [x_0]_\sim$, and so $[x_0]_\sim = \bigcup_{x \in [x_0]_\sim} C_x$, which is connected since each C_x has the shared point x_0. Thus every component is connected. ❦

[3.3.3] Remark

In a topological space, a component is referred to as a "maximal" connected subspace because it is completely saturated: any connected subspace belongs to exactly one component, and adjoining any outside element to a component would strictly break its connectedness.

An analogous process may be applied to split a space into path components.

[3.3.4] Definition (*Path Components*)

Suppose X is a topological space. Define $\sim$ to be an equivalence relation on X such that, if $x \sim y$, then there is some path from x to y contained in X. The equivalence classes for $\sim$ are called the **path components** of X.

Proof.

(1) **Reflexivity.** $x \sim x$ since $\gamma : [a, a] \to X$, where $\gamma(a) = x$, is a path from x to x.

(2) **Symmetry.** Immediate.

(3) **Transitivity.** Suppose $x \sim y$ and $y \sim z$. Then there are continuous maps $\gamma_1 : [a, b] \to X$ and $\gamma_2 : [b, c] \to X$ such that γ_1 parameterizes a path from x to y and γ_2 parameterizes a path from y to z. By the pasting lemma, note that $\gamma : [a, c] \to X$ defined by "pasting" γ_1, γ_2 is a path from x to z, and so $x \sim z$.

❦

[3.3.5] Theorem

The path components of X are path-connected, disjoint subspaces of X whose union is X. Every nonempty path-connected subspace of X is fully contained in one equivalence class (i.e., it intersects exactly one equivalence class).

Proof. The proof is identical to the proof about connected components. The proof for showing path components are path-connected is an immediate application of transitivity. ❦

[3.3.6] Remark

Note that the connected components C of a topological space X are closed. Note that if a subspace $A \subseteq X$ is connected, then so is its closure $\overline{A}$. Since all connected subspaces must be the subset of a connected component C, and the closure $\overline{C}$ of a connected component is connected, a connected component C equals its closure $\overline{C}$. Put another way, we established that connected components C are maximal, and so they must equal their closures. If there are finitely many connected components $C_1, \ldots, C_n$ of X, then each C_i is open, since the complement of one connected component C_i is the finite union of closed sets $\bigcup_{j \neq i} C_j$, which is closed.

To summarize, the connected components of a topological space are all closed, and if there are finitely many of them, they're all open.

[3.3.7] Definition (*Locally Connected, Locally Path Connected*)

Suppose X is a topological space. We say X is **locally connected at** x (resp. **locally path-connected at** x) if, for every neighborhood $U \subseteq X$ of x, there is some open, connected (resp. path connected) neighborhood $V \subseteq X$ of x fully contained in U. If all points in X are locally connected (resp. locally path connected), then we say X is locally connected (resp. locally path connected).

[3.3.8] Theorem

A space X is locally connected if and only if, for every open $U \subseteq X$, the connected components of U are open.

Proof.

($\Longrightarrow$): Suppose X is locally connected. Let $U \subseteq X$ be open, and let $C \subseteq U$ be some connected component of U. For each $x \in C$, there is some open, connected neighborhood V_x of x such that $x \in V_x \subseteq U$. Since $V_x \cap C \neq \emptyset$ and C is a connected component, it follows that $V_x \subseteq C$ since V_x is connected too. Thus $C = \bigcup_{x \in C} V_x$, the arbitrary union of open sets, which is open.

($\Longleftarrow$): Let $U \subseteq X$ be an arbitrary open subset. Then note, for each $x \in U$, there is some connected component $C \subseteq U$ such that $x \in C \subseteq U$. By hypothesis, C is open, completing the proof. ❦

[3.3.9] Theorem

A space X is locally path-connected if, for each open $U \subseteq X$, the path components of U are open.

Proof. The proof is identical to the proof of the previous theorem. ❦

[3.3.10] Theorem

If X is a topological space, then each path component of X is fully contained in one connected component of X. If X is locally path-connected, then the path components and connected components coincide.

> **Proof.** Let P be a path component of X. Since P is path-connected, it is also connected, so it is fully contained in some connected component $C \subseteq X$, proving the first statement.
>
> Now assume X is locally path-connected. To prove the second statement, we show $C \subseteq P$. For the sake of contradiction, suppose $P \subsetneq C$. Let Q be the union of all path components of X that are different from P and intersect C. Since all path components are contained in exactly one connected component, it follows that $Q \subseteq C$. Then note
>
> $$C = C \cap X = C \cap (P \cup P^c) = (C \cap P) \cup (C \cap P^c) = P \cup Q.$$
>
> Since every path component of a locally path-connected set is open, we have that P, Q are disjoint, nonempty open subsets whose union is C, meaning C is separated (a contradiction, since C is connected). Thus $C \subseteq P$, and so $P = C$.

3.3.1 Exercises

[3.3.11] Problem

What are the connected components and path components of $\mathbb{R}_\ell$? What are the continuous maps $f : \mathbb{R} \to \mathbb{R}_\ell$?

> **Proof.** Suppose $C \subseteq \mathbb{R}_\ell$ contains at least two distinct points, $x < y$. Then $C = \big((-\infty, y) \cap C\big) \cup \big([y, \infty) \cap C\big)$ constitutes a separation of C, so C is disconnected. Thus, the only nonempty connected subspaces of $\mathbb{R}_\ell$ are singletons, which are therefore the connected components. Since every path-connected space is connected, the path components must also be exactly the singletons.
>
> Note that for some $f : \mathbb{R} \to \mathbb{R}_\ell$ to be continuous, the image of a connected set must be connected. Since $\mathbb{R}$ is connected in its standard topology, it follows that $f(\mathbb{R})$ is connected, but the only connected subsets of $\mathbb{R}_\ell$ are singletons, so it follows that $f(\mathbb{R}) = \{*\}$, so all continuous maps are constant maps.

[3.3.12] Problem

(1) What are the connected components and path components of $\mathbb{R}^\omega$ in the product topology?

(2) Consider $\mathbb{R}^\omega$ in the uniform topology. Show that $\mathbf{x}, \mathbf{y}$ lie in the same component of $\mathbb{R}^\omega$ if and only if the sequence $(\mathbf{x}-\mathbf{y})$ is bounded above.

Proof.

(1) Note that the arbitrary product of connected and path-connected sets is connected and path-connected in the product topology. Thus $\mathbb{R}^\omega$ is the only connected and path-connected component in $\mathbb{R}^\omega$ with the product topology.

(2) $(\Longrightarrow)$: Suppose $\mathbf{x}, \mathbf{y}$ are in the same connected component. Then there is some connected $C' \subseteq \mathbb{R}^\omega$ such that $\mathbf{x}, \mathbf{y} \in C'$. Now consider a translation $T : \mathbb{R}^\omega \to \mathbb{R}^\omega$ defined by $T(\mathbf{x}) = \mathbf{x} - \mathbf{y}$. This is a homeomorphism, so $C := T(C')$ is also a connected component. Thus we have $\mathbf{x}-\mathbf{y}, \mathbf{0} \in C$.

First, we show $\mathbb{R}^\omega$ can be separated by the subspaces of bounded and unbounded sequences. Let $O \subseteq \mathbb{R}^\omega$ be the subspace of all bounded sequences. For each bounded sequence $x \in O$, define $B_x = B_{\overline{\rho}}(x, 1/2)$, a ball about x defined by the uniform metric of radius $1/2$. For each $y \in B_x$, note that $|x_n - y_n| < 1/2$ for every $n \in \mathbb{Z}_+$, and so y is bounded as well, meaning $y \in O$. Thus $x \in B_x \subseteq O$, so $O = \bigcup_{x \in O} B_x$. An analogous procedure can be performed to show O^c is open, since drawing a ball of radius $1/2$ about an unbounded sequence still lies in O^c. Thus $\mathbb{R}^\omega = O \cup O^c$, a separation.

We know $\mathbf{0}$ is bounded. If $\mathbf{x}-\mathbf{y}$ were unbounded, then $C = (O \cap C) \cup (O^c \cap C)$ would be a valid separation for a supposedly connected set (a contradiction), so $\mathbf{x}-\mathbf{y}$ is forced to be bounded.

$(\Longleftarrow)$: Suppose $\mathbf{x}-\mathbf{y}$ is a bounded sequence. Let $O \subseteq \mathbb{R}^\omega$ be the subspace of all bounded sequences. We will show O is connected, which will complete the proof.

We proceed by showing O is path-connected. Note the translation $T : \mathbb{R}^\omega \to \mathbb{R}^\omega$ given by $T(\mathbf{z}) = \mathbf{z} - \mathbf{y}$ is a homeomorphism. Since $\mathbf{x}-\mathbf{y}$ and $\mathbf{y}-\mathbf{y} = \mathbf{0}$ are both bounded sequences, we have $T(\mathbf{x}), T(\mathbf{y}) \in O$. We can show there is a path from $\mathbf{h} = \mathbf{x}-\mathbf{y}$ to $\mathbf{0}$ in O. We may define a map $\gamma : [0,1] \to O$ by $\gamma(t) = t\mathbf{h}$. Indeed, $\gamma(0) = \mathbf{0}$ and $\gamma(1) = \mathbf{h}$, so to show γ is a path, we simply show γ is continuous.

Since $\mathbf{h} \in O$, there is some $M \in \mathbb{R}$ such that $|h_n| < M$ for every $n \in \mathbb{Z}_+$. Fix some $t_0 \in [0,1]$ and some $1 > \varepsilon > 0$. Let $\delta = \varepsilon / M$, then let $t \in [0,1]$ be such that $|t - t_0| < \delta$. Then

$$\begin{aligned}
\overline{\rho}(\gamma(t), \gamma(t_0)) &= \sup_{n \in \mathbb{Z}_+} \{\min(|th_n - t_0 h_n|, 1)\} \\
&= \sup_{n \in \mathbb{Z}_+} \{|t - t_0||h_n|\} \\
&< \sup_{n \in \mathbb{Z}_+} \{(\varepsilon / M)(M)\} \\
&= \sup_{n \in \mathbb{Z}_+} \{\varepsilon\} = \varepsilon.
\end{aligned}$$

If $\varepsilon \geq 1$, then δ may be anything since the uniform metric is at most 1. Thus γ is continuous, and we conclude O is path-connected, which implies O is also connected. Because $T(\mathbf{x})$ and $T(\mathbf{y})$ both belong to the connected subspace O, they are in the same component. Since T^{-1} is a homeomorphism, it maps components to components, meaning $\mathbf{x}$ and $\mathbf{y}$ must also belong to the same component in $\mathbb{R}^\omega$, completing the proof.

[3.3.13] PROBLEM

Let X be locally path-connected. Show all open, connected subsets of X are path-connected.

> **Proof.** Let $O \subseteq X$ be open and connected. Fix $x \in O$ and any open neighborhood $U \subseteq O$ of x. Since U is an open subset of the locally path-connected space X, all of U's path components are open. Thus there is some open, path component (which is thus path-connected) $[x]_\sim$ such that $x \in [x]_\sim \subseteq U$. Thus O is locally path-connected, and so its path components and connected components coincide. Since O is connected, its only connected component is O, and so O is also path-connected, completing the proof. ❦

3.4 Compact Sets

[3.4.1] Definition (*Covering*)

A collection of subsets $\mathcal{A}$ of X is said to be a **cover** for X if the union of each set in $\mathcal{A}$ equals X. $\mathcal{A}$ is said to be an **open covering** if each element of $\mathcal{A}$ is an open subset of X.

Let $Y \subseteq X$. Then a collection of subsets $\mathcal{A}$ of X is said to be a cover for Y if Y is contained in the union of each set in $\mathcal{A}$.

[3.4.2] Definition (*Compact Set*)

Suppose X is a topological space. Then $C \subseteq X$ is said to be **compact** if every open cover of C has a finite subcover.

[3.4.3] Theorem (*Compactness of Subspace*)

Suppose X is a topological space, and $Y \subseteq X$ is a subspace. Then Y is compact if and only if, for every open cover of X that covers Y, there is a finite subcover that also covers Y.

Proof.

($\Longrightarrow$): Suppose $\mathcal{A}$ is a collection of open subsets of X that covers Y. Then note $\mathcal{A} \cap Y$, the intersection of each set in $\mathcal{A}$ by Y, forms an open cover of Y. Thus there is a finite subcover $\mathcal{B} \cap Y$ that covers Y. Then $\mathcal{B}$ is a finite collection of open subsets of X that covers Y.

($\Longleftarrow$): Let $\mathcal{B}$ be an open cover for Y. Each $B \in \mathcal{B}$ can be written as $B = A \cap Y$, where the set of all A's forms a collection of open subsets $\mathcal{A}$ of X that covers Y. There is a finite subcover of these that, when intersected with Y, forms a finite subcover for $\mathcal{B}$, completing the proof. ❦

[3.4.4] Theorem (*Closed Subspace of Compact Space is Compact*)

Any closed subspace of a compact space is compact.

Proof. Let X be a compact space, and let $Y \subseteq X$ be closed. Let $\mathcal{A}$ be an open cover of Y. Then note the collection defined by $\mathcal{B} = \mathcal{A} \cup \{X \setminus Y\}$ is an open cover of X ($X \setminus Y$ is open, since Y is closed). Since X is compact, there is a finite subcover B of X from $\mathcal{B}$. Then $B \setminus \{X \setminus Y\}$ is a finite subcover of $\mathcal{A}$, completing the proof. ❦

[3.4.5] Lemma (*Separation of a Point and a Compact Set*)

Suppose X is a Hausdorff space and $Y \subseteq X$ is compact. If $x_0 \notin Y$, then there are disjoint open neighborhoods containing x_0 and Y.

Proof. Fix $x_0 \in X \setminus Y$—we seek to find disjoint open neighborhoods containing x_0 and Y. For each $y \in Y$, it is obviously true that $x_0 \neq y$. Thus, by Hausdorffness of X, note that there are disjoint open neighborhoods $U_y \subseteq X$ of x_0 and $V_y \subseteq X$ of y. Then note $\{V_y\}_{y \in Y}$ forms a cover of Y by open sets from X, and by compactness of Y, we have that there exists a finite subcover $\{V_{y_k}\}_{k=1}^n$ of these open sets for Y. Thus we may write $Y \subseteq \bigcup_{k=1}^n V_{y_k}$.

Let $U = \bigcap_{k=1}^n U_{y_k}$ and $V = \bigcup_{k=1}^n V_{y_k}$, which are open subsets of X containing x_0 and Y, respectively. Then any $v \in V$ belongs to V_{y_k} for some $k \in \{1, 2, \ldots, n\}$. By how V_{y_k} was chosen, it follows that $v \notin U_{y_k}$, and so $v \notin U$. Thus, U and V are disjoint open neighborhoods containing x_0 and Y. ❦

[3.4.6] Theorem (*Compact Subspace of Hausdorff Space is Closed*)

Any compact subspace of a Hausdorff space is closed.

> **Proof.** Suppose X is a Hausdorff space, and let $Y \subseteq X$ be compact. We will show Y is closed by showing $X \setminus Y$ is open. Fix $x_0 \in X \setminus Y$. By the preceding lemma, note that there are disjoint open neighborhoods U and V containing x_0 and Y, respectively. Since $Y \subseteq V$ and U and V are disjoint, it follows that U cannot intersect Y. Thus $U \cap Y = \emptyset$, and so $U \subseteq X \setminus Y$. Thus $x_0 \in U \subseteq X \setminus Y$, implying $X \setminus Y$ is open and Y is closed. ❦

[3.4.7] Theorem (*Continuous Image of Compact Set is Compact*)

The continuous image of a compact set is compact.

> **Proof.** Suppose X is a compact set, and $f : X \to Y$ is continuous. Let $\mathcal{A}$ be a cover for $f(X)$ with open subsets of Y. Since the preimage of an open set is open, the preimage $f^{-1}(\mathcal{A})$ is an open cover of X. By compactness, there is a finite subcover $\{f^{-1}(A_k)\}_{k=1}^n$ with open sets of X. Then $f\left(\{f^{-1}(A_k)\}_{k=1}^n\right) = \{A_k\}_{k=1}^n$ forms a finite subcover of $f(X)$, completing the proof. ❦

[3.4.8] Theorem

Suppose $f : X \to Y$ is a bijective, continuous function. If X is compact and Y is Hausdorff, then f is a homeomorphism.

> **Proof.** To show f is a homeomorphism, we show f sends closed sets to closed sets. Let $O \subseteq X$ be closed, and thus consequently compact since all closed subsets of a compact space is compact. Thus $f(O) \subseteq Y$ is closed, and thus consequently closed since all compact subsets of a Hausdorff space are closed. Thus f sends closed sets to closed sets. ❦

[3.4.9] Lemma (*Tube Lemma*)

Suppose X, Y are topological spaces, with Y compact. Fix $x_0 \in X$. If $N \subseteq X \times Y$ is an open set containing $\{x_0\} \times Y$, then N contains some tube $W \times Y$ about $\{x_0\} \times Y$, where $W \subseteq X$ is an open neighborhood of x_0.

Proof. Fix $x_0 \in X$ and consider the set $\{x_0\} \times Y \subseteq X \times Y$. Let $N \subseteq X \times Y$ be an open neighborhood of $\{x_0\} \times Y$. For each $y \in Y$, choose some open neighborhoods U_y of x_0 and V_y of y such that $(x_0, y) \in U_y \times V_y \subseteq N$. Then $\{U_y \times V_y\}_{y \in Y}$ forms an open cover for $\{x_0\} \times Y$. Note that $\{x_0\} \times Y \cong Y$, and so the left-hand side is compact, and so there is a finite subcover $\{U_k \times V_k\}_{k=1}^n$ of $\{x_0\} \times Y$. Define

$$W = \bigcap_{k=1}^{n} U_k,$$

an open neighborhood in X about x_0. We will show $W \times Y \subseteq N$, completing the proof.

Let $(x, y) \in W \times Y$ be arbitrary. Then note $(x_0, y) \in \{x_0\} \times Y$, with y being the same as chosen from $W \times Y$. Thus there is some $j \in \{1, 2, \ldots, n\}$ such that $y \in V_j$, reason being that $\{U_k \times V_k\}_{k=1}^n$ is a finite subcover for $\{x_0\} \times Y$. Recall that $x \in U_j$ as well, since $x \in W$, and so $(x, y) \in (U_j \times V_j) \subseteq N$. Thus $W \times Y \subseteq N$, completing the proof. ❦

[3.4.10] Theorem (*Finite Product of Compact Spaces is Compact*)

The product of finitely many compact spaces is compact.

Proof. Suppose X, Y are compact spaces, and fix $x \in X$. Let $\mathcal{A}$ be an open cover for $X \times Y$—we show it has a finite subcover.

Consider the slice $\{x_0\} \times Y$: note that it is homeomorphic to the compact space Y, so there is a finite subcover $\{A_k\}_{k=1}^m$ for $\{x_0\} \times Y$. Let $N = \bigcup_{k=1}^m A_k$, an open set in $X \times Y$ containing $\{x_0\} \times Y$. By the Tube Lemma, there is some open neighborhood $W_x \subseteq X$ of x such that $W_x \times Y \subseteq N$. Thus $\{A_k\}_{k=1}^m$ forms a finite cover for $W_x \times Y$. Finally, note $\{W_x\}_{x \in X}$ is an open cover of X. By compactness of X, we have that $\{W_k\}_{k=1}^n$ forms a finite subcover for X. Thus we have that the union of the tubes

$$W_1 \times Y, W_2 \times Y, \ldots, W_n \times Y$$

is all of $X \times Y$; since each tube has a finite cover of open sets from $\mathcal{A}$, we have that $X \times Y$ is covered by finitely many sets from $\mathcal{A}$, completing the proof.

The proof for more compact sets proceeds by induction on the number of sets. ❦

We now investigate the finite intersection property and a reformulation of compact spaces in terms of them.

[3.4.11] Definition (*Finite Intersection Property*)

A collection of sets $\mathcal{C}$ is said to have the **finite intersection property** if, for every finite subcollection $\{C_k\}_{k=1}^n$ of $\mathcal{C}$, $\bigcap_{k=1}^n C_k \neq \emptyset$.

[3.4.12] Theorem (*Compactness in Terms of Finite Intersection Property*)

Let X be a topological space. Then X is compact if and only if, for every collection $\mathcal{C}$ of closed subsets of X that have the finite intersection property, the intersection $\bigcap_{C\in\mathcal{C}} C$ of all the elements of $\mathcal{C}$ is nonempty.

Proof. Given a collection $\mathcal{A}$ of subsets of X, we may define

$$\mathcal{C} = \{X \setminus A : A \in \mathcal{A}\}.$$

Then the following statements are true:

(1) $\mathcal{A}$ is a collection of open subsets of X if and only if $\mathcal{C}$ is a collection of closed subsets of X.

(2) $\mathcal{A}$ forms a cover for X if and only if $\bigcap_{C\in\mathcal{C}} C = \emptyset$.

(3) $\{A_k\}_{k=1}^n$ forms a finite subcover for X if and only if $\bigcap_{k=1}^n C_k = \emptyset$, where $C_k = X \setminus A_k$.

The first statement is by definition of a closed set, and the other two are a direct application of De Morgan's Laws. Namely that

$$X \setminus \left(\bigcup_{\lambda\in\Lambda} A_\lambda \right) = \bigcap_{\lambda\in\Lambda} (X \setminus A_\lambda).$$

Consider the original definition of compactness: X is compact if and only if, for any open cover $\mathcal{A}$, there is a finite subcover. Thus if $\mathcal{A}$ has no finite subcover for X, then it follows that $\mathcal{A}$ does not cover the compact set X. Now let $\mathcal{C} = \{X \setminus A : A \in \mathcal{A}\}$ as before. Immediately we see that compactness may be rephrased as such:

"*We say that X is compact if, for every collection of closed subsets $\mathcal{C}$ of X with the finite intersection property, the intersection of all elements in $\mathcal{C}$ is nonempty.*"

❦

[3.4.13] Remark (*Cantor's Intersection Theorem*)

Suppose $\{C_k\}_{k\in\mathbb{Z}_+}$ is a sequence of closed subsets of a compact space X, such that $C_k \supseteq C_{k+1}$ for each $k \in \mathbb{Z}_+$. Immediately we see that any finite subcollection of $\{C_k\}$ has nonempty intersection, meaning this sequence has the finite intersection property. By compactness, we have that $\bigcap_{C\in\mathcal{C}} C \neq \emptyset$. This result is familiar to us with $X = \mathbb{R}$.

[3.4.14] Recap

(1) For any topological space X, a collection of subsets $\mathcal{A}$ of X is said to **cover** X if $\bigcup_{A\in\mathcal{A}} A = X$. An **open cover** is a cover comprised of open sets. A subcover of X is a subset of $\mathcal{A}$ that still manages to cover X.

(2) A space X is said to be **compact** if, for every open cover $\mathcal{A}$, there is a finite subcover of X.

(3) For a subspace $Y \subseteq X$, a collection of subsets $\mathcal{A}$ of X is said to cover Y if $Y \subseteq \bigcup_{A\in\mathcal{A}} A$. This is different from the traditional definition of a cover for one space.

(4) A space $Y \subseteq X$ is compact if and only if, for every open cover of Y from open subsets of X, there is a finite subcover.

> **Intuition.** The idea behind this proof is that an open set in Y is an open set in X intersected with Y. Thus open covers can be moved around seamlessly between X and Y: simply move an open cover to the space that is given as compact by hypothesis, take the finite subcover, then move it back to the other space.

(5) Every closed subspace of a compact space is compact.

> **Intuition.** Let X be compact and $Y \subseteq X$ closed. Of course, $X \setminus Y$ is open. For any open cover of Y, we may adjoin $X \setminus Y$ to form an open cover for X, use its compactness, then simply get rid of the $X \setminus Y$ term to get a finite subcover of Y.

(6) Every compact subspace of a Hausdorff space is closed.

> **Intuition.** Let X be Hausdorff and $Y \subseteq X$ compact. We want to show $X \setminus Y$ is open. Fixing $x_0 \in X \setminus Y$, we show the set is open by showing there exists an open neighborhood $U \subseteq X$ of x_0 lying in $X \setminus Y$—that is, $U \cap Y = \emptyset$. For each $y \in Y$, we have that $y \neq x_0$, so by Hausdorffness there exist disjoint open neighborhoods $U_y \subseteq X$ of x_0 and $V_y \subseteq X$ of y. Then note $\{V_y\}_{y\in Y}$ forms an open cover for the compact space Y, so there exists a finite subcover $\{V_{y_k}\}_{k=1}^n$ of Y. Then define $U = \bigcap_{k=1}^n U_{y_k}$, an open neighborhood of x_0. For any $y \in Y$, there is some $k \in \{1,\dots,n\}$ such that $y \in V_{y_k}$, implying $y \notin U_{y_k}$, so $y \notin U$.

(7) We extract the key mechanism from the last proof: for any compact subspace Y of a Hausdorff space X, we can always construct disjoint open neighborhoods U, V about x_0, Y respectively—that is, $x_0 \in U \subseteq X \setminus Y$, meaning $X \setminus Y$ is open.

(8) The continuous image of a compact set is compact.

> **Intuition.** Let $f : X \to Y$ be a continuous map from the compact space X to a target space Y. Let $\mathcal{A}$ be an open covering for $f(X)$. Then note $f^{-1}(\mathcal{A})$ is an open covering of X by continuity of f. Since X is compact, we have a finite subcover $\{f^{-1}(A_k)\}_{k=1}^n$. Pushing forward yields that $f(X)$ is covered by the finite subcover $\{A_k\}_{k=1}^n$.

(9) Let $f : X \to Y$ be a bijective, continuous function. If X is compact and Y is Hausdorff, then f is a homeomorphism.

> **Intuition.** We show f^{-1} is continuous by showing f takes closed sets to closed sets. For some closed set $C \subseteq X$, we note C is also compact by a previous theorem. Since f is continuous, it follows that $f(C) \subseteq Y$ is compact in a Hausdorff space, which makes $f(C)$ closed.

(10) Let X, Y be topological spaces, with Y compact. For fixed $x_0 \in X$ and any open neighborhood $N \subseteq X \times Y$ of $\{x_0\} \times Y$, there is some open neighborhood $W \subseteq X$ of x_0 such that $W \times Y \subseteq N$.

> **Intuition.** Fix $x_0 \in X$ and let $N \subseteq X \times Y$ be an open neighborhood of $\{x_0\} \times Y$. Form a cover for $\{x_0\} \times Y$, of the form $\{U_y \times V_y\}_{y\in Y}$, where $U_y \times V_y$ is a basic, open neighborhood about (x_0, y). Note that $\{x_0\} \times Y \cong Y$, and since Y is compact, we form an open cover $\{U_k \times V_k\}_{k=1}^n$. Then we define $W = \bigcap_{k=1}^n U_k$, an open neighborhood of x_0. We show $\{U_k \times V_k\}_{k=1}^n$ covers not only $\{x_0\} \times Y$, but also $W \times Y$. Let $(x, y) \in W \times Y$ be arbitrary, and let $(x_0, y) \in \{x_0\} \times Y$ be the corresponding element with the same y-coordinate. Thus $y \in V_j$ for some $j \in \{1,\dots,n\}$, and since $x \in U_k$ for every k, $x \in U_j$. Thus $(x, y) \in (U_j \times V_j) \subseteq N$, completing the proof.

(11) The product of finitely many compact sets is compact.

> **Intuition.** We prove this for two compact topological spaces X, Y, and the full theorem is proven in a straightforward manner by induction. Suppose $\mathcal{A}$ is a cover of $X \times Y$. Fix $x \in X$, then note $\{x\} \times Y \cong Y$ is compact, so there is some finite subcover $\{A_k\}_{k=1}^m$ of sets from $\mathcal{A}$. Then $N = \bigcup_{k=1}^m A_k$ is an open subset of $X \times Y$ containing $\{x\} \times Y$. By the Tube Lemma, there is some open neighborhood $W_x \subseteq X$ of x such that $W_x \times Y \subseteq N$. Then $\{W_x\}_{x \in X}$ forms an open cover for X, and since X is compact, there is a finite subcover $\{W_k\}_{k=1}^n$. Thus
>
> $$\{W_1 \times Y, W_2 \times Y, \dots, W_n \times Y\}$$
>
> forms an open cover for $X \times Y$. Each element of this set also has a finite subcover of elements from $\mathcal{A}$, thus the entirety of $X \times Y$ has a finite subcover from elements of $\mathcal{A}$.

(12) A topological space X is compact if, for every collection of closed subsets $\mathcal{C}$ with the finite intersection property, the intersection $\bigcap_{C \in \mathcal{C}} C \neq \emptyset$.

> **Intuition.** Observe that, for any collection of subsets $\{\mathcal{A}_\lambda\}_{\lambda \in \Lambda}$ of X, it is true that
>
> $$X \setminus \left(\bigcup_{\lambda \in \Lambda} A_\lambda \right) = \bigcap_{\lambda \in \Lambda} (X \setminus A_\lambda).$$
>
> Defining $\mathcal{C} = \{X \setminus A : A \in \mathcal{A}\}$, where $\mathcal{A}$ is an open cover of X, and reformulating compactness in terms of $\mathcal{C}$ and the relation above yields this theorem.

3.4.1 Exercises

[3.4.15] Problem

(1) Let $\mathcal{T}, \mathcal{T}'$ be topologies on X such that $\mathcal{T} \subseteq \mathcal{T}'$. What does compactness of X in one topology say about the other?

(2) Show that if X is compact and Hausdorff under topologies $\mathcal{T}, \mathcal{T}'$, the topologies are either equal or incomparable.

Proof.

(1) If X is compact in $\mathcal{T}'$, then all open covers of X have finite subcovers—necessarily, this includes covers formed by sets in $\mathcal{T}$, and so X is compact in $\mathcal{T}$. No statement can be made in the reverse direction.

(2) Suppose the two topologies are comparable: without loss of generality, say $\mathcal{T} \subseteq \mathcal{T}'$. Then define the map

$$f : (X, \mathcal{T}') \to (X, \mathcal{T}) \quad f(x) = x.$$

Note f is a bijective, continuous map. Notably, it sends closed sets to closed sets: if $C \subseteq X$ is closed, then it is compact, and since f is continuous, $f(C)$ is also compact (and thus closed since the codomain is Hausdorff). Thus f is a homeomorphism and so $\mathcal{T} = \mathcal{T}'$. Thus the topologies are either equal or incomparable.

[3.4.16] Problem

Show that every compact subspace of a metric space is closed and bounded (with respect to the metric). Give an example of a closed and bounded subset of a metric space that is not necessarily compact.

Proof. Suppose (X, d) is a metric space in its metric topology, and suppose $C \subseteq X$ is compact. Note that the metric topology is Hausdorff, and since C is a compact subset of a Hausdorff space, it is closed. To show it is bounded, let $\mathcal{A}$ be an open cover of C with basic, open subsets from X. Then we may take a finite subcover $\{A_1, \ldots, A_n\}$. Finitely many open balls can be contained in exactly one ball of finite radius, which means C is bounded.

Consider the metric space $\mathbb{Q}$ together with metric $d(x, y) = |x - y|$. Then note that $[0, \sqrt{2}) \subseteq \mathbb{Q}$ is closed and bounded. Then consider the open cover $\mathcal{A} = (-1, 1) \cup \{(0, \sqrt{2} - 1/n)\}_{n \in \mathbb{Z}_+}$, which has no finite subcover.

[3.4.17] Problem

Suppose A, B are disjoint, compact subspaces of the Hausdorff space X. Show that there exist disjoint open sets U, V such that $A \subseteq U$ and $B \subseteq V$.

Proof. Note that in a Hausdorff space, a compact space and a point outside of it can be separated by two disjoint open neighborhoods. Namely, for each $x \in A$, there are disjoint open neighborhoods $U_x \subseteq X$ of x and $V_x \subseteq X$ of B. Then note $\{U_x\}_{x \in A}$ forms an open cover of A, and by compactness of A, we have a finite subcover $\{U_k\}_{k=1}^n$. Then simply define

$$U = \bigcup_{k=1}^n U_k \quad V = \bigcap_{k=1}^n V_k,$$

which are both open open sets containing A and B respectively. For each $u \in U$, there is some $j \in \{1, \ldots, n\}$ such that $u \in U_j$, meaning $u \notin V_j$ and so $u \notin V$. Thus U, V are disjoint, completing the proof.

3.5 Compact Subspaces of the Real Line

[3.5.1] Theorem (*Closed Intervals are Compact in Order Topology*)

Let X be a totally ordered set with the least upper bound property. In its order topology, closed intervals $[a,b] \subseteq X$ are compact subsets of X.

Proof. We split this proof into four steps. For setup, let $a < b \in X$, and consider the interval $[a,b]$ with an arbitrary open cover $\mathcal{A}$ of $[a,b]$, where the open sets are taken from the subspace topology on $[a,b]$ (which coincides with the order topology). We show there is a finite subcover of $\mathcal{A}$.

(1) First we show that, for any $x \in [a,b)$, there is some $y > x$ such that $[x,y]$ can be covered by, at most, two elements of $\mathcal{A}$. If y is the immediate successor to x, then note that $[x,y] = \{x,y\}$, so there is certainly one element of $\mathcal{A}$ that contains x and one element (not necessarily distinct) of $\mathcal{A}$ that contains y. Now suppose y is not the immediate successor to x. There is some $A \in \mathcal{A}$ such that $x \in A$. Since A is open, there is some $c \in [a,b]$ such that $x \in [x,c) \subseteq A$. Then choose $y \in (x,c)$, and so $[x,y] \subseteq A$.

(2) Let C be the set of points $y > a$ in $[a,b]$ such that $[a,y]$ can be covered by finitely many elements of $\mathcal{A}$. Then the set C is nonempty by (1), using $x = a$ (there is definitely some y such that $[a,y]$ can be covered by up to two elements of $\mathcal{A}$).

(3) Let $c = \sup(C)$—we seek to show $c \in C$. Let $A \in \mathcal{A}$ be some open set containing c; then there is some $d \in [a,b]$ such that $c \in (d,c] \subseteq A$. For the sake of contradiction, suppose $c \notin C$. Then there is some point z of C such that $z \in (d,c)$, since otherwise d would be a smaller upper bound for C. Since $z \in C$, there are finitely many sets from $\mathcal{A}$ that covers $[a,z]$, say n. Then $[z,c] \subseteq A$, so $[a,c] = [a,z] \cup [z,c]$ can be covered by, at most, $n+1$ sets, which is finite. Thus $c \in C$.

(4) Finally, we show $c = b$. Suppose, for the sake of contradiction, that $c < b$. Then, by (1), there is some $y \in (c,b]$ such that $[c,y]$ is covered by up to two sets from $\mathcal{A}$. Then $[a,y] = [a,c] \cup [c,y]$ can be covered by finitely many elements, so $y \in C$, contradicting that c is an upper bound for C.

Thus $[a,b]$ can be covered by finitely many elements from $\mathcal{A}$, completing the proof. ❦

[3.5.2] Remark (*Intuition Behind Proof*)

This proof is quite long, so we break it down intuitively. Consider any closed interval $[a,b]$ from the space X in its order topology with the least upper bound property. Let $\mathcal{A}$ be an open cover for $[a,b]$ comprising subsets of $[a,b]$ that are open in its subspace topology (which coincides with its order topology). First we establish that, for any point $x \in [a,b)$, there is some element of this interval next to x such that the area between them ($[x,y]$) can be covered by finitely many members of $\mathcal{A}$. Obviously we exclude b, since nothing comes after b. Knowing this, we know that there is some element after a such that the area between them can be covered by finitely many members of $\mathcal{A}$. We let C be all the elements of $[a,b]$ that satisfy this, and since it is nonempty, the value $c = \sup(C)$ is well-defined. For any member of $\mathcal{C}$ that contains c, since it is open, there is some nontrivial intersection with C, say at $z \in C$. By definition of C, we know $[a,z]$ can be finitely covered, and since $[z,c] \subseteq A$, the entire interval $[a,c] = [a,z] \cup [z,c]$ can be finitely covered, and so $c \in C$. Finally, to show $c = b$, we note that if $c < b$, then there is some $y \in (c,b]$ such that $[c,y]$ is finitely covered, and so $[a,y] = [a,c] \cup [c,y]$ is finitely covered, contradicting that c is an upper bound for C. Thus $c = b$, completing the proof.

[3.5.3] Corollary

Every closed interval in $\mathbb{R}$ is compact.

[3.5.4] Theorem (*Compact iff. Closed and Bounded*)

A subspace $A \subseteq \mathbb{R}^n$ is compact if and only if it is closed and bounded under either the square or Euclidean metric.

Proof. Note that $\rho(x,y) \le d(x,y) \le \sqrt{n}\rho(x,y)$ for $x,y \in \mathbb{R}^n$, so we only consider the square metric, since A is bounded under the Euclidean metric if and only if it is bounded under the square metric.

($\Longrightarrow$): Suppose A is compact. Since it is a compact subspace of a Hausdorff space, it is automatically closed, so we simply show A is bounded. Let $\{B_\rho(\mathbf{0}_n, m)\}_{m \in \mathbb{Z}_+}$ be an open cover for $\mathbb{R}^n$—then a finite subcover of this covers A. Then a box in $\mathbb{R}^n$ can contain the entirety of A, implying it is bounded.

($\Longleftarrow$): Suppose A is closed and bounded under ρ. Let $N \in \mathbb{R}$ be such that $\rho(x,y) \le N$ for every $x,y \in A$. Choose some $x_0 \in A$, then let $\rho(x_0, \mathbf{0}_n) = b$. Then, for any $x \in A$, note that

$$\rho(x, \mathbf{0}) \le \rho(x, x_0) + \rho(x_0, 0) = N + b.$$

Then A is a closed subset of the compact subspace $B_\rho(\mathbf{0}, N+b)$, implying A is compact. ❦

[3.5.5] Theorem (*Extreme Value Theorem*)

Suppose $f : X \to Y$ is a continuous map from a compact set X to an ordered set Y in its order topology. Then there are $c, d \in X$ such that $f(c) \le f(x) \le f(d)$ for every $x \in X$.

Proof. Consider $A = \operatorname{im}(f) = f(X)$, a compact set by continuity. We show that A contains a maximum M and minimum m, which completes the proof when taking $c \in f^{-1}(\{m\})$ and $d \in f^{-1}(\{M\})$. We show that A contains a maximum and omit the proof for a minimum, as it is entirely symmetric.

For the sake of contradiction, suppose A has no maximum. Then note $\mathcal{A} = \{(-\infty, a)\}_{a \in A}$ is a collection of open subsets of X. For any $y \in A$, note there is some $a \in A$ such that $y < a$, and so $y \in (-\infty, a)$. Thus $\mathcal{A}$ forms an open cover for $f(X)$.Take a finite subcover $\{(-\infty, a_k)\}_{k=1}^n$ with each $a_k \in A$. Choosing $a = \max\{a_k\}_{k=1}^n$, note that $a \in A$ but a is not covered by this finite subcover, a contradiction. Thus A has a maximum. ❦

For $X, Y = \mathbb{R}$, note this is the Extreme Value Theorem from calculus.

We now prove a theorem about uniform continuity: that a continuous map from a compact metric space to a target metric space is uniformly continuous. In the process, we will use something known as a Lebesgue number and the Lebesgue covering lemma.

[3.5.6] Definition (*Distance from Point to Metric Space*)

Suppose (X,d) is a metric space, and let $A \subseteq X$ is nonempty. Then, for any $x \in X$, we define

$$d(x,A) = \inf_{a \in A}\{d(x,a)\}.$$

Generally, $d(x,A)$ is the "shortest" distance from a point to any member of a set A. Contrast this with the *diameter* of A, which is defined as the "largest" distance between any two points in A (that is, the diameter is $\sup_{x,y \in A}\{d(x,y)\}$).

[3.5.7] Theorem

For a metric space (X,d) and a fixed, nonempty subset $A \subseteq X$, the function $d(-,A)$ is continuous.

Proof. Fix $x,y \in X$. Then note

$$d(x,A) \leq d(x,a) \leq d(x,y) + d(y,a)$$

for each $a \in A$. It follows that

$$d(x,A) - d(x,y) \leq \inf_{a \in A}\{d(y,a)\} = d(y,A) \implies d(x,A) - d(y,A) \leq d(x,y).$$

Switching the roles of x,y, we arrive at the inequality

$$|d(x,A) - d(y,A)| \leq d(x,y).$$

For any $\varepsilon > 0$, choosing $\delta = \varepsilon$ proves $d(-,A)$ is continuous by the $\varepsilon - \delta$ characterization of continuity. ❦

[3.5.8] Theorem (*Lebesgue Covering Lemma*)

Let (X,d) be a compact metric space together with any arbitrary open cover $\mathcal{A}$. Then there exists some $\delta > 0$, called a **Lebesgue number**, such that every nonempty subset of X with diameter less than δ is contained in some member of the cover.

Proof. Let (X,d) be a compact metric space, and let $\mathcal{A}$ be an open cover of X. Then take a finite subcover $\{A_k\}_{k=1}^n$. For each $k \in \{1,\dots,n\}$, define $C_k = X \setminus A_k$. Then define a function

$$f : X \to \mathbb{R} \quad f(x) = \frac{1}{n}\sum_{k=1}^{n} d(x,C_k).$$

We show $f(x) > 0$ for every $x \in X$. For fixed $x \in X$, there is some k such that $x \in A_k$. Since A_k is open, we may choose some $\varepsilon > 0$ such that $x \in B(x,\varepsilon) \subseteq A_k$. Then $d(x,C_k) \geq \varepsilon$, and so $f(x) \geq \varepsilon/n$.

Note f is a constant multiple of a sum of continuous functions, and so f is continuous. Thus f attains a minimum value δ, which is greater than 0 by the previous statement, by the Extreme Value Theorem, which we will show is our desired Lebesgue number.

Let $B \subseteq X$ be a nonempty subset whose diameter is less than δ. Choose some $x_0 \in B$, then note B lies in the δ-neighborhood of x_0. Thus

$$\delta \leq f(x_0) \leq d(x_0, C_m),$$

where C_m is chosen such that it is the maximum of $d(x_0,C_k)$ for each k. Then the δ-neighborhood of x_0 is contained in the element $A_m = X \setminus C_m$ of the covering $\mathcal{A}$, meaning B is as well. ❦

Recall the definition of uniform continuity.

[3.5.9] Definition (*Uniform Continuity*)

Suppose (X, d_X) and (Y, d_Y) are metric spaces. A map $f : X \to Y$ is said to be **uniformly continuous** if, for every $\varepsilon > 0$, there exists some $\delta > 0$ such that for every $x_1, x_2 \in X$, it follows that

$$d_X(x_1, x_2) < \delta \implies d_Y(f(x_1), f(x_2)) < \varepsilon.$$

[3.5.10] Theorem (*Uniform Continuity Theorem*)

Suppose $f : X \to Y$ is a continuous map from the compact metric space (X, d_X) to a target metric space (Y, d_Y). Then f is uniformly continuous.

> **Proof.** Fix $\varepsilon > 0$. Let $\mathcal{A} = \{B_{d_Y}(y, \varepsilon/2)\}_{y \in Y}$ be an open cover for Y, and consider the inverse image $f^{-1}(\mathcal{A})$, which forms an open cover for X. Since X forms a compact metric space, there is a Lebesgue number $\delta > 0$. For any set $\{x_1, x_2\}$ whose diameter is less than δ, there is some $A \in \mathcal{A}$ such that $\{x_1, x_2\} \subseteq A$. Note $A = f^{-1}(B_{d_Y}(y, \varepsilon/2))$ for some $y \in Y$, and so $f(x_1), f(x_2) \in B_{d_Y}(y, \varepsilon/2)$. Thus $d_Y(f(x_1), f(x_2)) < \varepsilon$ for any pair $x_1, x_2 \in X$, completing the proof. ❦

We now prove that the reals are uncountable. Namely, we use the order properties of $\mathbb{R}$, avoiding decimals and binary expansions and the like.

[3.5.11] Definition (*Isolated Point*)

Let X be a topological space. A point $x \in X$ is said to be **isolated** if $\{x\}$ is open. Equivalently, x is isolated if it is not a limit point of X, or if there exists an open neighborhood $O \subseteq X$ of x that has no other points.

[3.5.12] Theorem

Suppose X is a nonempty, compact, Hausdorff space. If X has no isolated points, then it is uncountable.

> **Proof.** First, we claim that for any nonempty open $U \subseteq X$ and $x \in X$, there is a nonempty open $V \subseteq U$ such that $x \notin \overline{V}$. Choose $y \in U$ distinct from x. (If $x \in U$, y exists because x is not isolated; if $x \notin U$, y exists because U is nonempty.) Since X is Hausdorff, there are disjoint open neighborhoods W_1 and W_2 of x and y. Let $V = W_2 \cap U$. Then $V \subseteq U$ is nonempty and open (since $y \in V$). Since W_1 is an open neighborhood of x disjoint from V, $x \notin \overline{V}$.
>
> To show X is uncountable, let $f : \mathbb{Z}_+ \to X$ be any map with $f(n) = x_n$. We show f is not surjective. For $n = 1$, applying the claim to $U = X$ allows us to choose a nonempty open $V_1 \subseteq X$ such that $x_1 \notin \overline{V}_1$. Inductively, given V_{n-1}, applying the claim to $U = V_{n-1}$ allows us to choose a nonempty open $V_n \subseteq V_{n-1}$ such that $x_n \notin \overline{V}_n$.
>
> Thus, $\overline{V}_1 \supseteq \overline{V}_2 \supseteq \cdots$ is a sequence of nonempty closed sets with the finite intersection property. By compactness, $\bigcap_{n \in \mathbb{Z}_+} \overline{V}_n$ is nonempty, so it contains some element x. Since $x_n \notin \overline{V}_n$ for all n, $x \neq x_n$. Thus $x \notin f(\mathbb{Z}_+)$, meaning f is not surjective and X is uncountable. ❦

[3.5.13] Corollary

Every nonempty compact subset of $\mathbb{R}$ is uncountable.

[3.5.14] Recap

(1) Let X be a totally ordered set with the least upper bound property. In its order topology, closed intervals of X are compact subspaces.

> **Intuition.** Let $[a,b] \subseteq X$ be a closed interval, and let $\mathcal{A}$ be an open cover for this interval, where it takes open subsets of $[a,b]$ in its subspace topology. First, we show that for every $x \in [a,b)$, there is some $y > x$ such that $[x,y] \subseteq [a,b]$ can be covered by finitely many members of the cover $\mathcal{A}$ (in fact, by up to two!).
>
> Then we consider starting at a. Let C be the set of all points $y > a$ such that $[a,y] \subseteq [a,b]$ can be covered by finitely many members of $\mathcal{A}$. We are guaranteed that $C \neq \emptyset$, since letting $x = a$ and applying the previous logic proves there is some element in C. Since it is non-empty and bounded, there is some $c = \sup(C)$. Let $A \in \mathcal{A}$ be a member of the cover such that $c \in A$. By A being open, it follows there is some $d < c$ such that $c \in (d,c] \subseteq A$. Note that there is some $z \in C$ such that $d < z < c$, otherwise d would be a smaller upper bound than C. Then $[a,c] = [a,z] \cup [z,c]$, where the first part is covered by finitely many members of $\mathcal{A}$ as per the definition of C, and the second part is covered by finitely many members of $\mathcal{A}$ since $[z,c] \subseteq A$—thus $c \in C$.
>
> Now we show $c = b$. For the sake of contradiction, suppose $c < b$. Following the logic of the previous paragraph, there is some $A \in \mathcal{A}$ such that $c \in A$, and since A is open, there is some $x \in (c,b]$ such that $[c,x] \subseteq A$. Then $[a,x] = [a,c] \cup [c,x]$ can be finitely covered, meaning $x \in C$, contradicting that c is an upper bound. Thus $c = b$.

(2) As a consequence, taking $X = \mathbb{R}$ means are closed intervals in $\mathbb{R}$ are compact.

(3) **Compact iff. Closed and Bounded in $\mathbb{R}^n$.** A subspace $A \subseteq \mathbb{R}^n$ is compact if and only if it is closed and bounded under the Euclidean or square metrics.

> **Intuition.** Note that a subspace $A \subseteq \mathbb{R}^n$ is bounded under the Euclidean metric if and only if it is bounded under the square metric, since
> $$\rho(x,y) \leq d(x,y) \leq \sqrt{n}\rho(x,y)$$
> for every $x, y \in \mathbb{R}^n$.
>
> ($\Longrightarrow$): Suppose A is compact. Then A is a compact subspace of the Hausdorff space $\mathbb{R}^n$, and so A is closed. Let $\{B_\rho(\mathbf{0}_n, m)\}_{m \in \mathbb{Z}_+}$ be an open cover for $\mathbb{R}^n$, then choose a finite subcover that covers A. Then the union of all members of the finite subcover is contained in a box, and since A is inside this box, A is bounded as well.
>
> ($\Longleftarrow$): Suppose A is closed and bounded under ρ. Suppose that $N \in \mathbb{R}$ is such that $\rho(x,y) \leq N$ for each $x, y \in A$. Fix some $x_0 \in A$, then let $\rho(x_0) = b$. Then note $\rho(x, \mathbf{0}_n) \leq \rho(x, x_0) + \rho(x_0, \mathbf{0}_n) = N + b$ for every $x \in A$, and so $A \subseteq B_\rho(\mathbf{0}_n, N+b)$. Since A is a closed subset of the compact box, A is compact as well.

(4) **Extreme Value Theorem.** Let $f : X \to Y$ be a continuous map from a topological space X to a totally ordered set Y in its order topology. If X is compact, there exist $c, d \in X$ such that $f(c) \leq f(x) \leq f(d)$ for every $x \in X$.

> **Intuition.** This theorem essentially states that $f(X)$ has a minimum and maximum. We prove $f(X)$ contains its maximum, since the proof that it contains its minimum is analogous. For the sake of contradiction, we suppose $f(X)$ does not contain a maximum. Then note $\{(-\infty, a)\}_{a \in f(X)}$ is a set of open subsets of Y. Since $f(X)$ has no maximum, for each $y \in f(X)$, there is some $a > y \in f(X)$, so this collection forms an open cover for $f(X)$. Since X is compact, $f(X)$ is also compact by continuity of f. Thus there is a finite subcover
> $$\{(-\infty, a_1), \ldots, (-\infty, a_n)\}$$
> that covers $f(X)$. Note $a = \max\{a_1, \ldots, a_n\}$ exists and should be an element of A, but it is not contained in the finite subcover, a contradiction.

(5) For a metric space (X,d) and some nonempty subset $A \subseteq X$, we define the distance of a point $x \in X$ from A by $d(x,A) = \inf_{a \in A}\{d(x,a)\}$. Colloquially, this is the shortest distance from the point to the set A. Compare this to the diameter of A, which is colloquially the largest distance between any two points in A.

(6) The function $d(-,A)$ is continuous.

> **Intuition.** Let $x, y \in X$ be arbitrary points. Note then that
> $$d(x,A) \le d(x,a) \le d(x,y) + d(y,a)$$
> for every $a \in A$. Notably, we seek to minimize $d(y,a)$, such that
> $$d(x,A) \le d(x,y) - d(y,a) \le d(x,y) - \inf_{a\in A}\{d(y,a)\} \implies d(x,A) - d(y,A) \le d(x,y).$$
> Interchanging the role of x, y yields $|d(x,A) - d(y,A)| \le d(x,y)$, implying $d(-,A)$ is Lipschitz and thus continuous.

(7) **Lebesgue Covering Lemma.** Suppose (X,d) is a compact metric space together with an open cover $\mathcal{A}$. Then there exists some $\delta > 0$, called a **Lebesgue number**, such that every set whose diameter is less than δ is contained in one member of the cover.

> **Intuition.** Since X is compact, it admits a finite subcover $\{A_k\}_{k=1}^n$, where each $A_k \in \mathcal{A}$. Let $C_k = X \setminus A_k$ for each k, then define the function $f : X \to \mathbb{R}$ by
> $$f(x) = \frac{1}{n}\sum_{k=1}^{n} d(x, C_k).$$
> We demonstrate that $f(x) > 0$ for every $x \in X$. For a fixed $x \in X$, there is some k such that $x \in A_k$. Since A_k is open, there is some $\varepsilon > 0$ such that $x \in B(x,\varepsilon) \subseteq A_k$. Thus $d(x, C_k) \ge \varepsilon$ and so $f(x) \ge \varepsilon/n$. Since X is compact, f attains a minimum value $\delta > 0$ (where $\delta > 0$ since $f(x) > 0$) by EVT. We show this is our desired Lebesgue number.
>
> Consider any set B whose diameter is δ. Then there is some $x_0 \in B$ such that $B \subseteq B(x_0, \delta)$. Then note
> $$\delta \le f(x_0) \le d(x, C_m),$$
> where m is chosen to maximize the value of $d(x, C_k)$ for each k. Then $B(x_0,\delta) \subseteq A_m$, where $A_m = X \setminus C_m$, implying $B \subseteq A_m \in \mathcal{A}$, completing the proof.

(8) **Uniform Continuity Theorem.** Suppose $f : X \to Y$ is a continuous map from a compact metric space (X, d_X) to a target metric space (Y, d_Y). Then f is uniformly continuous, meaning that for any $\varepsilon > 0$, there is some $\delta > 0$ such that
$$d_X(x_1, x_2) < \delta \implies d_Y(f(x_1), f(x_2)) < \varepsilon$$
for every $x_1, x_2 \in X$.

> **Intuition.** Fix $\varepsilon > 0$. Let $\mathcal{A} = \{B_{d_Y}(y, \varepsilon/2)\}_{y\in Y}$ be an open cover of $f(X)$ from open subsets of Y. Then note $f^{-1}(\mathcal{A})$ forms an open cover of X. Since X forms a compact metric space, there is a Lebesgue number δ. Let $x_1, x_2 \in X$ be such that $d_X(x_1, x_2) < \delta$. Then $\{x_1, x_2\}$ is a set whose diameter is less than δ, meaning $\{x_1, x_2\} \subseteq A$ for some $A \in \mathcal{A}$. This implies $\{x_1, x_2\} \subseteq f^{-1}(B_{d_Y}(y, \varepsilon/2))$ for some $y \in Y$, and so $\{f(x_1), f(x_2)\} \subseteq B_{d_Y}(y, \varepsilon/2)$. Thus $d_Y(f(x_1), f(x_2)) < \varepsilon$, completing the proof.

(9) For any topological space X, a point $x \in X$ is said to be **isolated** if $\{x\}$ is open. Equivalently, x is isolated if an open neighborhood of x exists that comprises only x, or if x is not a limit point of X.

(10) Suppose X is a nonempty, compact, Hausdorff space. If X has no isolated points, then X is uncountable.

Intuition. The intuition behind this proof is to construct a sequence of nested, open sets $V_1 \supseteq V_2 \supseteq \cdots$ with the finite intersection property, such that $x_n \in V_n$ but $x_n \notin \overline{V}_{n+1}$ for every $n \in \mathbb{Z}_+$, where $x_n \in X$. Then, taking the closure of each V_k (note that if $V_k \subseteq V_{k+1}$, then $\overline{V}_k \subseteq \overline{V}_{k+1}$), we yield a sequence of nested, closed sets with the finite intersection property. Thus $x \in \bigcup_{k \in \mathbb{Z}_+} \overline{V}_k$, but $x \neq x_n$ for each n, completing the proof that X is uncountable.

Let X be a nonempty, compact, Hausdorff space with no isolated points. Let $U \subseteq X$ be a nonempty, open set and let $x \in X$ be arbitrary. We seek to show there is a nonempty, open subset $V \subseteq U$ such that $x \notin \overline{V}$. Choose a point $y \in U$ such that $x \neq y$—if $x \in U$, then U necessarily contains a distinct point y to prevent $\{x\}$ from being open (since X has no isolated points); otherwise if $x \notin U$, then y is guaranteed since U is nonempty. By Hausdorffness of X, there exists open neighborhoods $W_1, W_2 \subseteq X$ of x and y repsectively. Then $V = W_2 \cap U$ is a nonempty, open subset of U. Note $x \notin \overline{V}$ since W_1 is an open neighborhood of X that doesn't intersect W_2, so x is not a limit point of V.

Consider the subset $\{x_n\}_{n \in \mathbb{Z}_+} \subseteq X$. Let $V_1 = X$, and let $V_{k+1} \subseteq V_k$ be nonempty and open such that $x \notin V_{k+1}$ (existence is proved by the last paragraph). Then the closure of each V_k forms a nested sequence of closed sets with the finite intersection property, so there is some $x \in \cup_{k \in \mathbb{Z}_+} \overline{V}_k$. Note that, for each $n \in \mathbb{Z}_+$, $x_n \notin \overline{V}_n$ and so $x \neq x_n$. Thus x is an element of X outside this subset, making X uncountable.

(11) As a direct corollary, it follows that $\mathbb{R}$ is uncountable.

3.5.1 Exercises

[3.5.15] Problem

Suppose X is a totally ordered set in its order topology. Show that if every closed interval of X is compact, then X has the least upper bound property.

Proof. Let $A \subseteq X$ be an arbitrary nonempty, bounded set. Let $a \in A$ be arbitrary, and let b be an arbitrary upper bound. Then consider the interval $[a, b] \subseteq X$. For the sake of contradiction, suppose A has no supremum. Then define an open cover for X by assigning a set to each point $x \in [a, b]$ and forming a collection of sets from it:

(1) If x is not an upper bound: choose some $a' \in A$ such that $x < a'$, then choose the open set $(-\infty, a')$.

(2) If x is an upper bound: choose some upper bound b' such that $b' < x$, then choose the open set (b', ∞).

Note these steps are justified since A has no supremum (i.e., $x \in [a, b]$ cannot be the largest element of A or the least upper bound). Then there is a finite subcover whose union contains $[a, b]$: that is,

$$[a, b] \subseteq \left[\bigcup_{k=1}^{m} (-\infty, a_k)\right] \cup \left[\bigcup_{k=1}^{n} (b_k, \infty)\right]$$

for some $a_k \in A$ for each $k \in \{1, \ldots, m\}$ and for some upper bound b_k of A for each $k \in \{1, \ldots, n\}$. We have that $b_{\min} = \min\{b_k\}_{k=1}^{n}$ is the smallest upper bound of A contained in $[a, b]$, and by convexity of $[a, b]$, we note $b_{\min}$ is the supremum for A, a contradiction. ❦

[3.5.16] Problem

Let (X, d) be a metric space, and let $A \subseteq X$ be nonempty.

(1) Show $d(x, A) = 0$ if and only if $x \in \overline{A}$.

(2) Suppose A is compact. Show that $d(x, A) = d(x, a)$ for some $a \in A$.

(3) Define the ε-neighborhood of A in X to be the set

$$U(A, \varepsilon) = \{x \in X : d(x, A) < \varepsilon\}.$$

Show that $U(A, \varepsilon) = \bigcup_{a \in A} B(a, \varepsilon)$.

Proof.

(1) ($\Longrightarrow$): Suppose $d(x, A) = 0$, meaning $\inf_{a \in A}\{d(x, a)\} = 0$. Let $\varepsilon > 0$ be arbitrary, then consider the open neighborhood $B(x, \varepsilon)$ of x. Since $d(x, A) = 0$, there is some $a \in A$ such that $d(x, a) < \varepsilon$. Thus $a \in B(x, \varepsilon)$, meaning $B(x, \varepsilon) \cap A \neq \emptyset$. Since every arbitrary basic open neighborhood of x intersects A, it follows that $x \in \overline{A}$.

($\Longleftarrow$): Suppose $x \in \overline{A}$. Fix some $n \in \mathbb{Z}_+$, then let $\varepsilon = 1/n$. Note then that $B(x, \varepsilon) \cap A \neq \emptyset$ by definition of $\overline{A}$, so there is some $a_n \in B(x, \varepsilon) \cap A$. Thus $d(x, a_n) < \varepsilon$, and as we let $n \to \infty$, we note $\varepsilon \to 0$. Thus $d(x, A) = 0$.

(2) Define $f : A \to \mathbb{R}$ by $f(a) = d(x, a)$, with $x \in X$ fixed. Let $a_1, a_2 \in A$ be arbitrary, then note

$$d(x, a_2) \leq d(x, a_1) + d(a_1, a_2) \implies d(x, a_2) - d(x, a_1) \leq d(a_1, a_2).$$

Reversing the roles of a_1, a_2, we arrive at $|d(x, a_2) - d(x, a_1)| \leq d(a_1, a_2)$, implying $d(x, -)$ is Lipschitz and thus continuous. Since f is continuous and A is compact, we have that $f(A) = \{d(x, a) : a \in A\}$ takes on a minimum value by the Extreme Value Theorem. Thus there is some $a \in A$ such that $f(a) = d(x, a) = \inf_{a \in A}\{d(x, a)\} = d(x, A)$.

(3) Suppose $x \in U(A, \varepsilon)$, and so $d(x, A) < \varepsilon$. Then there is some $a \in A$ such that $d(x, a) < \varepsilon$, and so $x \in B_d(a, \varepsilon) \subseteq \bigcup_{a \in A} B_d(a, \varepsilon)$. Conversely, suppose $x \in \bigcup_{a \in A} B_d(a, \varepsilon)$. Then, for some $a \in A$, we have that $d(x, A) \leq d(x, a) < \varepsilon$, so $x \in U(A, \varepsilon)$.

[3.5.17] Problem

Suppose (X, d) is a connected metric space. Show that if X has more than one point, then X is uncountable.

Proof. For fixed $x \in X$, define $f : X \to \mathbb{R}$ by $f(y) = d(x, y)$. We have that f is continuous, and since X is connected, the set $f(X)$ is connected as well. Note connected subsets of $\mathbb{R}$ coincide with intervals or rays, which are uncountable if they have at least one element, and so $f(X)$ is uncountable (since X has at least one element). Note that $|X| \geq |f(X)|$, so X is uncountable as well.

3.6 Limit Point Compactness

[3.6.1] Definition (*Limit Point Compactness*)

A space X is said to be **limit point compact** if every infinite subset of X has a limit point in X.

[3.6.2] Theorem (*Compactness $\implies$ Limit Point Compactness*)

A compact space is limit point compact.

Proof. Suppose X is compact. We seek to show if $A \subseteq X$ is infinite, then it contains a limit point in X. We prove by contrapositive: that if A has no limit points, then it is finite. Indeed, take A to have no limit points. Then A is trivially closed, since it contains all its limit points. Since A is a closed subset of a compact space, we have that A is compact as well. For each $a \in A$, note that $U_a = \{a\}$ forms an open neighborhood of a in A, and so $\{U_a\}_{a\in A}$ is an open cover for A. Since A is compact, we have that $\{U_k\}_{k=1}^n$ forms a finite subcover, meaning A has finitely many elements. ❦

[3.6.3] Example (*Limit Point Compact Set That Isn't Compact*)

Note the converse is not true in general. Consider the set $Y = \{a, b\}$ with the indiscrete topology. Then consider the set

$$X = \mathbb{Z}_+ \times Y = \bigcup_{n\in\mathbb{Z}_+} \{(n,a),(n,b)\}.$$

Let $U \subseteq X$ be nonempty. If U has any any integer paired with either a or b, then the same integer paired with the other element is a limit point. Since all nonempty subsets of X have a limit point, we have that X is limit point compact. However, note the open covering $\{\{n\} \times Y\}_{n\in\mathbb{Z}_+}$ for X has no finite subcover, and so X is not compact.

[3.6.4] Definition (*Sequential Compactness*)

A topological space X is said to be **sequentially compact** if, for every sequence $(x_n) \subseteq X$, there is some subsequence (x_{n_k}) that converges in X.

In the context of metric spaces, we note that compactness, limit point compactness, and sequential compactness all coincide.

[3.6.5] Theorem (*Compactness $\iff$ Limit Point Compactness $\iff$ Sequential Compactness in Metric Spaces*)

Suppose X is a metrizable topological space. Then the following are equivalent.

(1) X is compact.

(2) X is limit point compact.

(3) X is sequentially compact.

Proof. (1) $\implies$ (2): We have proven this already.

(2) $\implies$ (3): Suppose X is limit point compact. Let x_n be an arbitrary sequence in X, and let $A = \{x_n\}_{n \in \mathbb{Z}_+}$. If A is finite, then there is one number that repeats a countably infinite number of times in the sequence, so choose this as a subsequence and note it converges. Otherwise, if A is infinite, then by hypothesis we have a limit point x. For each $k \in \mathbb{Z}_+$, let n_k both be greater than all n_j for $j < k$ and be such that $x_{n_k} \in B(x, 1/k)$. Then $(x_{n_k}) \to x$, and so X is sequentially compact.

(3) $\implies$ (1): Suppose X is sequentially compact. We prove that the Lebesgue covering lemma applies to X, then we prove that finitely many ε-balls can cover X, then we finally prove X is compact.

Let $\mathcal{A}$ be an arbitrary open cover of X. For the sake of contradiction, suppose the Lebesgue covering lemma doesn't apply to X. Then, for each $n \in \mathbb{Z}_+$, there exists a nonempty set C_n whose diameter is less than $1/n$ and does not lie in a single member of $\mathcal{A}$. Choosing $x_n \in C_n$ for each n and forming a sequence, we use our hypothesis to get a convergent subsequence $(x_{n_k}) \to a$. Note that a belongs to some member A of the open cover $\mathcal{A}$, and since A is open, there is some $\varepsilon > 0$ such that $B(a, \varepsilon) \subseteq A$. We may choose an index i large enough such that $1/n_i < \varepsilon/2$, and since $\operatorname{diam}(C_{n_i}) < \varepsilon/2$, we have that $C_{n_i} \subseteq B(x, \varepsilon/2)$; moreover, we may choose i large enough such that $d(x_{n_i}, a) < \varepsilon/2$. For any point $c \in C_{n_i}$, note that

$$d(c, a) \leq d(c, x) + d(x, a) < \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

and so $C_{n_i} \subseteq B(a, \varepsilon)$. Thus $C_{n_i} \subseteq A$, contrary to assumption.

Suppose, for the sake of contradiction, that X cannot be covered by finitely many ε-balls, for a fixed $\varepsilon > 0$. Let $x_1 \in X$, then note $B(x_1, \varepsilon) \neq X$, otherwise X could be covered by finitely (namely one) ε-ball. Thus choose $x_2 \in X \setminus B(x_1, \varepsilon)$, and note $B(x_1, \varepsilon) \cup B(x_2, \varepsilon) \neq X$ for the same reason as before. In general, given distinct points $x_1, x_2, \ldots, x_n$ of X, choose x_{n+1} such that $x_{n+1} \notin \bigcup_{k=1}^{n} B(x_k, \varepsilon)$. By construction, we have that $d(x_{n+1}, x_i) \geq \varepsilon$ for every $i \in \{1, \ldots, n\}$, so there is no chance of (x_n) having a convergent subsequence, contrary to the fact that X is sequentially compact.

Finally, we show X is compact. Let $\mathcal{A}$ be an arbitrary open cover ofr X. Since X is sequentially compact, it has a Lebesgue number δ. Choose $\varepsilon = \delta/3$, then note that we may cover X with finitely many ε-balls. Each of these balls has a diameter of $2\delta/3$, so each ball fits in one member of the open cover. There are finitely members of the cover that contain all the balls, which cover X, and so X is compact. ❦

[3.6.6] Recap

(1) A set X is limit point compact if, for every infinite subset $A \subseteq X$, A has a limit point in X.

(2) If X is compact, then X is limit point compact.

> **Intuition.** Suppose X is compact, and let $A \subseteq X$ be an arbitrary subset. To prove X is limit point compact, we need to show that if A is infinite, then it has a limit point. We proceed by contrapositive, so suppose A has no limit points. Vacuously, A contains all its limit points, so A is closed. Since A is a closed subset of a compact space, A is also compact. Note that A has no limit points, so all its elements are isolated, meaning $U_a = \{a\}$ forms an open cover for A for each $a \in A$. Since A is compact, there is a finite subcover $\{U_k\}_{k=1}^n$ comprised of singletons for A, so A is finite (i.e., not infinite).

(3) Limit point compactness does not imply compactness in general. Consider $Y = \{a, b\}$ with the indiscrete topology and $X = \mathbb{Z}_+ \times Y$. Then X is limit point compact (each $\{n, a\}$ is a limit point of $\{n, b\}$ and vice versa), but it is not compact (consider the open cover $\{\{n\} \times Y\}_{n \in \mathbb{Z}_+}$ for X, and note it has no finite subcover).

(4) A set X is **sequentially compact** if every sequence in X has a convergent subsequence.

(5) In a metrizable space, compactness, limit point compactness, and sequential compactness are all equivalent.

> **Intuition.** Our proof of compactness implying limit point compactness is done.
>
> To show limit point compactness implies sequential compactness, we take some sequence $(x_n) \subseteq X$ and consider $A = \{x_n\}_{n \in \mathbb{Z}_+}$. If A is finite, then one term in the sequence repeats countably infinite amount of times, so a constant subsequence made from it converges, of course. Otherwise, if A is finite, it has a limit point a. For each $k \in \mathbb{Z}_+$, we may define an increasing sequence of indices n_k such that $x_{n_k} \in B(a, 1/k)$, and so $(x_{n_k}) \to a$.
>
> Suppose X is sequentially compact.
>
> (a) We first show X admits the Lebesgue covering lemma by contradiction. If it didn't, then—given a cover $\mathcal{A}$—we could define a sequence of nonempty sets $C_n \subseteq X$ such that $\operatorname{diam}(C_n) < 1/n$ and C_n doesn't lie in a single member of the cover. We may define a sequence (x_n) such that $x_n \in C_n$ for each $n \in \mathbb{Z}_+$, then note there is a convergent subsequence $(x_{n_k}) \to a$. Then a lies in a single member A of the open cover $\mathcal{A}$, and since A is open, there is some $\varepsilon > 0$ such that $a \in B(a, \varepsilon) \subseteq A$. Choose an index i large enough such that $1/n_i < \varepsilon/2$ (and so $\operatorname{diam}(C_{n_i}) < \varepsilon/2$, meaning $C_{n_i} \subseteq B(x, \varepsilon/2)$), and choose i large enough so that $d(x_{n_i}, a) < \varepsilon/2$. Then, for any $c \in C_{n_i}$, we have that
>
> $$d(c, a) \leq d(c, x) + d(x, a) < \varepsilon/2 + \varepsilon/2 = \varepsilon,$$
>
> so $C_{n_i} \subseteq B(a, \varepsilon) \subseteq A$, a contradiction.
>
> (b) Next, we show that X can be covered by finitely many ε-balls by contradiction. Fix $\varepsilon > 0$, and suppose the result is false. Choose $x_1 \in X$, then note $B(x_1, \varepsilon) \neq X$ (since finitely many ε-balls may not cover X), so there is some $x_2 \in X \setminus B(x_1, \varepsilon)$. Proceed inductively: that is, for distinct points $x_1, \ldots, x_n \in X$, there is some x_{n+1} such that $x_{n+1} \notin \bigcup_{k=1}^n B(x_n, \varepsilon)$. Then $d(x_{n+1}, x_i) \geq \varepsilon$ for every $i \in \{1, \ldots, n\}$, and so (x_n) has no convergent subsequences, a contradiction.
>
> (c) Finally, we show X is compact. Let $\mathcal{A}$ be an open cover for X, and choose a Lebesgue number $\delta > 0$. Let $\varepsilon = \delta/3$, then note there are finitely many ε-balls that cover X. Each ε-ball has diameter $2\delta/3 < \delta$, and so each belongs to one member of $\mathcal{A}$. Since there are finitely many of them, we choose a finite subcover of $\mathcal{A}$ that encompasses all the finitely many ε-balls that cover X, completing the proof.

3.6.1 Exercises

[3.6.7] PROBLEM

Show that $[0,1]^\omega$ equipped with the uniform topology is not limit point compact.

Proof. We seek to find some infinite subset $A \subseteq [0,1]^\omega$ with no limit points. Consider the set of all standard basis vectors of $\mathbb{R}^\omega$—that is, the set $A = \{e_n\}_{n\in\mathbb{Z}_+}$, where $e_n : \mathbb{Z}_+ \to [0,1]$ is defined by $e_n(j) = \delta_{n,j}$. Then, for any e_j, e_k for $j \neq k \in \mathbb{Z}_+$, we have that

$$\overline{\rho}(e_j, e_k) = \sup_{n\in\mathbb{Z}_+} |x_n - y_n| = 1.$$

For the sake of contradiction, suppose $x \in [0,1]^\omega$ is a limit point. Then $B(x,1/3) \cap [A \setminus \{x\}] \neq \emptyset$. In fact, $B(x,1/3)$ intersects A infinitely many times outside of x. If $B(x,1/3) \cap [A \setminus \{x\}] = \{e_1, \ldots, e_n\}$, then we could take $r = \min\{d(x,e_k)\}_{k=1}^n$ and choose $r' < r$ so that $B(x,r') \cap [A \setminus \{x\}] = \emptyset$ (and so x not a limit point), a contradiction. Thus let $e_1, e_2 \in B(x,1/3) \cap [A \setminus \{x\}]$. Then note

$$\overline{\rho}(e_1, x) \geq \overline{\rho}(e_1, e_2) - \overline{\rho}(x, e_2) = 1 - \underbrace{\overline{\rho}(x, e_2)}_{\leq 1/3} \geq 2/3,$$

which means $e_1 \notin B(x,1/3)$, a contradiction. ❦

[3.6.8] PROBLEM

Suppose X is limit point compact.

(1) Suppose $f : X \to Y$ is continuous. Is $f(X)$ limit point compact?

(2) If $A \subseteq X$ is closed, is A limit point compact?

(3) If X is a subspace of a Hausdorff space Z, is X closed?

Proof.

(1) No. Let $X = \mathbb{Z}_+ \times Y$, where $Y = \{a, b\}$ with the indiscrete topology. Then define $f : X \to \mathbb{Z}_+$ (where $\mathbb{Z}_+$ takes its subspace topology from $\mathbb{R}$) by $(n, *) \mapsto n$. Note that any singleton subset $\{n\} \subseteq \mathbb{Z}_+$ is a basic, open set, so showing $f^{-1}(\{n\})$ is open guarantees continuity. Indeed, the inverse image is $\{n\} \times Y$, which is an open subset of X. Of course, $\mathbb{Z}_+$ is not limit point compact in its metric topology, completing the disproof.

(2) Yes. Suppose $U \subseteq A$ is infinite. Since $U \subseteq X$, we have that U has a limit point $x \in X$. Since A contains all its limit points, it follows that $x \in A$, and so A is limit point compact.

(3) No. For our counterexample, we want A to have a limit point in $Z \setminus X$, meaning X is not closed (it does not contain all of its limit points). Consider $X = S_\Omega$, which is limit point compact, and $Z = \overline{S}_\Omega$, which is Hausdorff (all order topologies are Hausdorff). Note the subset $X \subseteq X$ has a limit point Ω, but $\Omega \notin X$, and so X is not closed. ❦

[3.6.9] Problem

A space X is said to be **countably compact** if, for any countable family of open sets that cover X, a finite subcover exists. Show that if X is T_1, then X is countably compact if and only if it is limit point compact.

Proof.

($\Longrightarrow$): Suppose X is T_1 and countably compact: we proceed by contrapositive. Suppose $A \subseteq X$ has no limit points: we show that A is finite. Indeed, note that A contains all of its limit points, and so it is closed and contains only isolated points. Thus singleton subsets $\{*\} \subseteq A$ are open in A. For the sake of contradiction, suppose A is infinite. Then we may choose a countably infinite set of distinct points $[A' = \{x_n\}_{n \in \mathbb{Z}_+}] \subseteq A$. Consider the countably infinite cover of X formed by taking each singleton subset of A' and adjoining it to $X \setminus A'$. By countable compactness of X, we take a finite subcover; discarding the $X \setminus A'$ element, we get a covering of A by finitely many singletons, a contradiction. Thus A is finite, meaning X is limit point compact.

($\Longleftarrow$): Suppose X is T_1 and limit point compact: we show X is countably compact by contradiction. Suppose X is not countably compact, and take a countable open cover $\mathcal{U} = \{U_n\}_{n \in \mathbb{Z}_+}$ of X. Note that there is no finite subcover, so for each $n \in \mathbb{Z}_+$, choose $x_n \notin U_1 \cup \cdots \cup U_n$. Since $X = \bigcup U_n$, the set $A = \{x_n\}_{n \in \mathbb{Z}_+}$ is infinite. By limit point compactness, A has a limit point x. Since $\mathcal{U}$ covers X, $x \in U_k$ for some $k \in \mathbb{Z}_+$. Since X is T_1, the neighborhood U_k of the limit point x must contain infinitely many points of A. However, by construction, $x_n \notin U_k$ for all $n \geq k$, implying $U_k \cap A \subseteq \{x_1, \ldots, x_{k-1}\}$. This finite intersection contradicts x being a limit point. Thus, X is countably compact.

[3.6.10] Problem

Suppose (X,d) is a metric space. If $f: X \to X$ is such that

$$d(f(x), f(y)) = d(x,y),$$

we say f is an **isometry**.

(1) Show that f is an embedding.

(2) If X is compact, show that f is a homeomorphism.

Proof.

(1) Suppose $f(x) = f(y)$ for some $x, y \in X$. Then note $0 = d(f(x), f(y)) = d(x,y)$, meaning $x = y$ and so f is injective. Next, choose $\varepsilon > 0$, and consider the ball $O = B(x, \varepsilon) \subseteq X$, a basic open set. For any $y \in O$, note that $d(f(x), f(y)) = d(x,y)$, so $f(y) \in B(f(x), \varepsilon)$. Indeed, $f(O) = B(f(x), \varepsilon) \cap f(X)$, meaning $f^{-1}|_{f(X)}$ is continuous. Showing $f: X \to f(X)$ is continuous follows almost exactly the same idea. Thus f is a homeomorphism onto its image, i.e. is an embedding.

(2) By continuity of f, we have that $f(X)$ is compact. Note that $f(X) \subseteq X$ is a compact subspace of a Hausdorff space, and so $f(X)$ is closed. For the sake of contradiction, suppose $a \in X \setminus f(X)$. Since a belongs to an open set, there is some $\varepsilon > 0$ such that $B(a, \varepsilon) \cap f(X) = \emptyset$. Note then that $d(a, f(a)) \geq \varepsilon$. Moreover, $d(f(a), f(f(a))) = d(a, f(a)) \geq \varepsilon$, where the equality is by hypothesis. Indeed, for $x_1 = a$ and $x_{n+1} = f(x_n)$, we have that $d(x_m, x_n) \geq \varepsilon$ for every $m \neq n$. Note the sequence $\{x_n\}_{n \in \mathbb{Z}_+ \setminus \{1\}} \subseteq f(X)$ has no convergent subsequence, and so $f(X)$ is not sequentially compact, a contradiction (since compactness and sequential compactness are equivalent in metrizable spaces). Thus $X \setminus f(X) = \emptyset$, and so $f(X) = X$, meaning f is a surjective embedding and thus a homeomorphism.

[3.6.11] Problem

Let (X,d) be a metric space. If $f: X \to X$ satisfies the condition

$$d(f(x), f(y)) < d(x,y) \quad \forall x \neq y \in X,$$

then f is a **shrinking map**. If there is a number $C < 1$ such that

$$d(f(x), f(y)) \leq C d(x,y) \quad \forall x \neq y \in X,$$

then f is a **contraction**. A **fixed point** of f is a point x^* such that $f(x^*) = x^*$.

(1) If $f: X \to X$ is a contraction with X compact, show f has a unique fixed point.

(2) If $f: X \to X$ is a shrinking map with X compact, show f has a unique fixed point.

Proof.

(1) Let $A_n = f^n(X)$, where f^n denotes applying the function n times. Note that $f: X \to X$, and so $f(X) \subseteq X$ since $f(X)$ is a subset of the codomain. We will show $f(f(X)) \subseteq f(X)$, and then by induction, it will follow that $A_{n+1} \subseteq A_n$ for any $n \in \mathbb{Z}_+$. If $y \in f(f(X))$, there is some $x \in f^{-1}(\{y\}) \in f(X) \subseteq X$, and so $y = f(x) \in f(X)$ as desired.

Note that contractions are Lipschitz and thus continuous. Since X is compact, each A_n is compact, and as a subspace of a metric (thus Hausdorff) space, each A_n is closed. Since A_n forms a sequence of nested closed sets, by compactness of X we have that there is some $x^* \in \bigcap_{n \in \mathbb{Z}_+} A_n$. We show x^* is our desired fixed point.

Let $A = \bigcap_{n \in \mathbb{Z}_+} A_n$, and note $x^* \in A$. Note that all potential fixed points must live in A, so to show x^* is unique, we show A has only one point. Since $A_{n+1} \subseteq A_n$, we have that $\operatorname{diam}(A_{n+1}) \leq \operatorname{diam}(A_n)$. Since X is a compact metric space, we have that X is bounded, meaning $\operatorname{diam}(X) = M$ for some $M \in \mathbb{R}$. Then $\operatorname{diam}(A_n) = \operatorname{diam}(f^n(X)) \leq C^n M$, and as $n \to \infty$, we have $\operatorname{diam}(A_n) \to 0$, implying $\operatorname{diam}(A) = 0$. Thus $A = \{x^*\}$.

Finally, to show x^* is a fixed point, we show $f(A) \subseteq A$. If $y \in f(A)$, then $y = f(x^*)$. Since $x^* \in A_n$ for each $n \in \mathbb{Z}_+$, we have that $f(x^*) \in f(A_n) = A_{n+1} \subseteq A_n$ for each $n \in \mathbb{Z}_+$. Thus $y \in A$, meaning $f(\{x^*\}) \subseteq \{x^*\}$. This forces $f(x^*) = x^*$, meaning x^* is a fixed point of f, and by the previous argument, this fixed point is unique.

(2) Let $A_n = f^n(X)$ and $A = \bigcap_{n \in \mathbb{Z}_+} A_n$. As in the previous part, f is continuous (since it is a shrinking map, it is 1-Lipschitz), each A_n is closed and nested, and A is a nonempty compact set with $f(A) \subseteq A$. We show $A \subseteq f(A)$.

Let $x \in A$. For each $n \in \mathbb{Z}_+$, since $x \in A_{n+1} = f^{n+1}(X)$, there exists $x_n \in X$ such that $x = f^{n+1}(x_n)$. Define $y_n = f^n(x_n)$, noting $y_n \in A_n$ and $f(y_n) = x$ for all n. Since X is sequentially compact, (y_n) has a convergent subsequence $(y_{n_k}) \to a \in X$. For any fixed $m \in \mathbb{Z}_+$, the tail of the subsequence lies entirely in the closed set A_m, so $a \in A_m$. Thus $a \in A$. By continuity of f, $f(a) = \lim_{k \to \infty} f(y_{n_k}) = \lim_{k \to \infty} x = x$. Thus $x \in f(A)$, so $A = f(A)$.

We now show $\operatorname{diam}(A) = 0$. Since A is compact, if $\operatorname{diam}(A) > 0$, the continuous function d attains a maximum on $A \times A$, so there exist $p, q \in A$ such that $d(p,q) = \operatorname{diam}(A)$. Since $A = f(A)$, we can choose $u, v \in A$ such that $f(u) = p$ and $f(v) = q$. Since $p \neq q$, we must have $u \neq v$. Because f is a shrinking map,

$$\operatorname{diam}(A) = d(p,q) = d(f(u), f(v)) < d(u,v) \leq \operatorname{diam}(A),$$

which is a contradiction. Thus $\operatorname{diam}(A) = 0$, meaning $A = \{x^*\}$.

Since $f(A) = A$, we have $f(x^*) = x^*$. If y is another fixed point, $d(x^*, y) = d(f(x^*), f(y)) < d(x^*, y)$, a contradiction. Thus x^* is unique.

3.7 Local Compactness

[3.7.1] Definition (*Local Compactness*)

A space X is said to be **locally compact at** x if there exists some compact set $C \subseteq X$ that contains an open neighborhood of x. If each point of X is locally compact, X is said to be **locally compact**.

[3.7.2] Example (*Examples of Locally Compact Spaces*)

(1) Any compact space X is locally compact, since for each $x \in X$, there is some open neighborhood O of x such that $x \in O \subseteq X$, with X being the compact superspace.

(2) $\mathbb{R}$ is locally compact. For each $x \in \mathbb{R}$, there is some $a, b \in \mathbb{R}$ such that $x \in (a, b) \subseteq [a, b]$, where $[a, b]$ is compact. The same is true of $\mathbb{R}^n$, where instead we can draw an open ball around a point x that is contained inside a closed one.

(3) Any ordered set X is compact since every basis element is contained in a closed interval.

(4) $\mathbb{R}^\omega$ in its product topology is *not* locally compact, since none of its basis elements are contained in compact superspaces. Take an arbitrary basic, open set $U = (\prod_{k=1}^n [a_k, b_k]) \times \mathbb{R} \times \cdots$. If U were contained in a compact set, then its closure $\overline{U}$ would be compact as well, since closed subsets of compact spaces are compact. However, $\overline{U}$ is the finite product of closed intervals, which is also multiplied by infinitely many instances of $\mathbb{R}$, making it not compct.

Of course, there are weird and arcane topological spaces. One method of demystifying such spaces is by considering them as subspaces of nice topological spaces, in hopes that it will inherit some nice properties. For a "bad" space X, we can consider an embedding $f : X \to Y$ into a "good" space Y, such that $(X \cong f(X)) \subseteq Y$. In reality, $f(X)$ is the true subspace of Y, but since X is homeomorphic to $f(X)$, X has the nice properties $f(X)$ would inherit as a subspace of Y. Through this, we are able to study a bad space by observing it as a subspace of a good space.

So what constitutes a good space? The examples we will consider are metrizable spaces and compact Hausdorff spaces. Metrizable spaces are nice due to a rigid topology defined by their metric, as well as topological notions (such as limit points and closure) being able to be probed by sequences. Compact Hausdorff spaces are nice since infinite covers can be reduced to finite subcovers, so some properties are more tractable, as well as sequences having unique limit points and other goodies.

Subspaces of metrizable spaces are metrizable as well, so no new information can be gathered from this. While subspaces of compact Hausdorff spaces are Hausdorff, they do not need to be compact. For example, consider the compact Hausdorff space $[0, 1]$ and a subspace $(0, 1)$, which is obviously not compact. In later sections we will discuss a more complete characterization of *every* space that can be embedded into a compact Hausdorff space, but for now we investigate a simpler question. We investigate what classes of spaces are such that adding one specific point can produce a compact Hausdorff space. We will see that it is indeed locally compact Hausdorff spaces: there is a special point one could adjoin to a locally compact Hausdorff space to produce a compact Hausdorff space. This process is called **one-point compactification**.

[3.7.3] Theorem (*Characterization of Locally Compact Hausdorff Spaces*)

Let X be a space. Then X is locally compact Hausdorff if and only if there exists a space Y such that:

(1) X is a subspace of Y.

(2) $Y \setminus X$ is a singleton.

(3) Y is compact Hausdorff.

The identified space Y is unique up to homeomorphism. If there are two spaces Y, Y' satisfying the aforementioned properties, then there is a homeomorphism from Y to Y' that fixes X.

Proof. We first verify uniqueness. Let X be a LCH space, and suppose Y, Y' are superspaces of X that have the prescribed properties, in such a way that $Y = X \cup \{\infty\}$ and $Y' = X \cup \{\infty'\}$. Define a map

$$h : Y \to Y' \quad h(x) = \begin{cases} \infty' & x = \infty, \\ x & \text{otherwise,} \end{cases}$$

a map that fixes X and takes $\infty \mapsto \infty'$. We show that h is a homeomorphism.

Obviously, h is a bijection. We show that h maps open sets to open sets, and the reverse is true by symmetry. To get started, suppose $O \subseteq Y$ is open. If $\infty \notin O$, then $O \subseteq X$ is open, and so $h(O) = O \subseteq X$ is also open. If $\infty \in O$, then note that $C = Y \setminus O$ is a closed subset of X, and so C is compact (since closed subspaces of Hausdorff spaces are compact). Since $X \subseteq Y'$, we have that C is a compact, and thus closed, subspace of Y'. Thus $h(O) = Y' \setminus C$ is open, making h a homeomorphism.

Now we show that if X is a LCH space, there is some point $\infty \notin X$ such that $Y = X \cup \{\infty\}$ is compact Hausdorff. We equip Y with the topology comprised of all sets U that are open subsets of X and all sets $Y \setminus C$ such that C is compact in X. We need to show that this topology is valid, the topology on X is the subspace topology from Y, and that Y is compact Hausdorff with this topology. The first two conditions are trivial to check.

We show Y is compact: consider an open cover $\mathcal{A}$ of Y. Note that there is some compact $C \subseteq X$ such that $Y \setminus C \in \mathcal{A}$, since open sets in X do not contain ∞. Then note all other members of $\mathcal{A}$, when intersected with A, form an open cover for C with open subsets of X. Since C is compact, a finite subcollection covers C, and when adjoined to the open set $Y \setminus C$, a finite subcollection of $\mathcal{A}$ covers Y. Thus Y is compact.

We show Y is Hausdorff; consider $x \neq y \in Y$. Separated neighborhoods exist if $x, y \in X$ since X is Hausdorff, so WLOG suppose $y = \infty$. Since X is locally compact, there is a compact $C \subseteq X$ such that there is some open neighborhood $U \subseteq C$ of x. Then $U, Y \setminus C$ are disjoint open neighborhoods of x and y respectively, and so Y is Hausdorff.

Now we prove the converse: if such a space Y exists that satisfies the following conditions, then X is locally compact (Hausdorffness is immediate). Given $x \in X$, we show X is locally compact at x. Choose disjoint open sets U, V of Y containing x and ∞. Then $C = Y \setminus V$ is closed in Hausdorff space Y and thus compact, and U is an open neighborhood of x that lies in the compact space $C \subseteq X$. Thus X is locally compact. ❦

If X happens to be a compact space, then the $Y = X \cup \{\infty\}$ from the preceding theorem is not interesting. Note if X is compact, we need that $Y \setminus X$ is open in our prescribed Y, so the singleton $\{\infty\}$ is open, making ∞ an isolated point. If X is not compact, then $Y \setminus X$ is not open, and so $\{\infty\}$ is not open. Thus since every open neighborhood of ∞ intersects X nontrivially, making ∞ a limit point of X. Thus $\overline{X} = Y$ for X not compact, where the closure is relative to Y. We call this process **compactification**.

[3.7.4] Definition (*Compactification, One-Point Compactification*)

If Y is a compact Hausdorff space and X is a proper subspace of Y such that $\overline{X} = Y$, Y is called the **compactification** of X. If $Y \setminus X$ is a singleton, then it is called the **one-point compactification**.

[3.7.5] Example (*Common Examples of Compactifications*)

Ultimately, we have shown that if X is LCH and not compact, then it admits a one-point compactification. Note this means that $\mathbb{R}$ admits the one-point compactification $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$, which can be readily seen to be homeomorphic to $\mathbb{R}^1$ by stereographic projection. The same is true for $\mathbb{R}^2$ and in general $\mathbb{R}^n$ with S^n. Note if we see $\mathbb{C} \cong \mathbb{R}^2$, we get a one-point compactification $\mathbb{C} \cup \{\infty\} \cong S^2$, and we even call the one-point compactification of $\mathbb{C}$ the *Riemann sphere*.

Note there is an alternative, more satisfying way to define local compactness. Typically, when a property is local about some $x \in X$, it means that there exists a neighborhood about x where the property is true. But our definition of local compactness is not framed in this way. Indeed, we will formulate an alternative definition that identifies behavior more local in nature. Note this characterization is only equivalent in the context of Hausdorff spaces, however.

[3.7.6] Theorem (*Characterization of Local Compactness*)

Let X be a Hausdorff space. Then X is locally compact if and only if, for any $x \in X$ and an open neighborhood U of x, there is some open neighborhood $V \subseteq U$ of x such that $\overline{V}$ is compact and $\overline{V} \subseteq U$.

> **Proof.**
>
> ($\Longrightarrow$): Suppose X is LCH. Let $x \in X$ and an open neighborhood $U \subseteq X$ of x be arbitrary. Let Y be the one-point compactification of X, then note $C = Y \setminus U$ is closed, and thus compact, in the Hausdorff space Y. Note that a compact subset of a Hausdorff space can be separated from a singleton outside the compact subset through open sets. Thus choose disjoint open sets V, W such that $x \in V$ and $C \subseteq W$. Thus $\overline{V}$ is closed and thus compact in the Hausdorff space Y, and since it is disjoint from C, we have $\overline{V} \subseteq U$ as desired.
>
> ($\Longleftarrow$): Suppose this formulation is true. Given $x \in X$ and an open neighborhood $U \subseteq X$, there exists a neighborhood V of x such that the compact space $\overline{V}$ is contained in U. Since $\overline{V}$ is a compact subspace containing the open neighborhood V of x, it follows that X is locally compact. ❦

That is, a Hausdorff space X is locally compact if and only if every neighborhood about any point $x \in X$ contains a compact subspace that contains an open neighborhood about x.

[3.7.7] Corollary (*Locally Compact Subspaces of LCH Spaces*)

Let X be an LCH space. If $A \subseteq X$ is open or closed, then A is locally compact.

> **Proof.** Suppose A is closed, and let $x \in A$. Note that X is locally compact, so there is some compact set $C \subseteq X$ that contains an open neighborhood $U \subseteq X$ of x. Then note $C \cap A$ is closed in C and thus compact, and it contains the open neighborhood $U \cap A$ of x in A.
>
> Now suppose A is open, and let $x \in A$. Since X is locally compact and A is a neighborhood of x, there is some neighborhood $V \subseteq A$ of x such that $\overline{V} \subseteq A$ is compact. Then $\overline{V}$ is a compact space containing the open neighborhood V of x, and so A is locally compact. ❦

[3.7.8] Theorem (*Characterization of Open Subspaces of Compact Hausdorff Space*)

A space X is homeomorphic to an open subspace of a compact Hausdorff space if and only if X is locally compact Hausdorff.

3.8 Nets

[3.8.1] Definition (*Partially Ordered Set, Directed Set*)

A **partially ordered set** is a set A together with a partial order $\leq$ satisfying the following conditions.

(1) **Reflexivity.** $a \leq a$ for every $a \in A$.

(2) **Symmetry.** For every pair $a, b \in A$, if $a \leq b$ and $b \leq a$, then $a = b$.

(3) **Transitivty.** For every triple $a, b, c \in A$, if $a \leq b$ and $b \leq c$, then $a \leq c$.

A **directed set** is some partially ordered set Λ such that for any pair $a, b \in \Lambda$, there is some $c \in \Lambda$ such that $a \leq c$ and $b \leq c$. That is, any pair of elements in Λ have an upper bound in the set.

[3.8.2] Definition (*Cofinal Sets*)

A subset K of a directed set Λ is **cofinal** if, for any $a \in \Lambda$, there is some $b \in K$ such that $a \leq b$.

[3.8.3] Remark (*Intuition behind Cofinality*)

Cofinality is precisely the algebraic translation of "marching off to infinity." For a standard sequence indexed by $\mathbb{Z}_+$, a subset of indices goes to infinity because it eventually exceeds any given integer N. For an arbitrary directed set Λ, a subset is cofinal if it eventually exceeds any given element $\alpha \in \Lambda$.

[3.8.4] Theorem

Cofinal subsets of a directed set are directed sets.

Proof. Let K be a cofinal subset of the directed set Λ. For any $a, b \in K$, note there is some $c \in \Lambda$ such that $a \leq c$ and $b \leq c$. Since K is cofinal, there is some $k \in K$ such that $c \leq k$. Thus $a, b \leq k$, and so K is a directed set. ❦

[3.8.5] Definition (*Net*)

Let X be a topological space. A **net** in X is a tuple of X indexed by Λ. That is, a net $\mathbf{x}$ is a map $f : \Lambda \to X$ such that $\mathbf{x} = (x_\lambda)_{\lambda \in \Lambda}$, where $x_\lambda = f(\lambda)$ for each $\lambda \in \Lambda$.

[3.8.6] Definition (*Convergence of Net*)

A net $(x_\lambda)_{\lambda \in \Lambda}$ is said to **converge** to a point x if, for any open neighborhood U of x, there is some $\lambda \in \Lambda$ such that $x_{\lambda'} \in U$ for every $\lambda' \geq \lambda$.

Many theorems about sequences can be lifted to nets in a completely identical way. We repeat them here anyways.

[3.8.7] Theorem (*Hausdorff Guarantees Unique Limit Point*)

Suppose X is a Hausdorff space. Then a convergent net $\mathbf{x} = (x_\lambda)_{\lambda \in \Lambda}$ has a unique limit.

Proof. For the sake of contradiction, suppose $\mathbf{x} \to x$ and $\mathbf{x} \to y$ for $x \neq y \in X$. Construct disjoint open neighborhoods U_x and U_y of x and y respectively. Then note the tail of $\mathbf{x}$ cannot lie in U_x and U_y simultaneously, since they are disjoint, which is a contradiction. Thus nets converge to, at most, one value. ❦

[3.8.8] Theorem (*Nets Probe Elements of Closure*)

Let X be a topological space with $A \subseteq X$. Then $x \in \overline{A}$ if and only if a net of points from A converges to x.

Proof.

($\Longrightarrow$): Suppose $x \in \overline{A}$. Let Λ be the collection of all open neighborhoods of x, directed by reverse inclusion so that $U_\lambda \subseteq U_{\lambda'}$ for every $\lambda \geq \lambda'$ (where $U_\lambda = \lambda$). Since $x \in \overline{A}$, there is some $x_\lambda \in U_\lambda \cap A$ for each $\lambda \in \Lambda$. Define the net $\mathbf{x} = (x_\lambda)_{\lambda \in \Lambda}$. For any open neighborhood U of x, note that there is some $\lambda \in \Lambda$ such that $U_\lambda \subseteq U$ (specifically, choosing $\lambda = U$). Thus, for each $\lambda' \geq \lambda$, we have that $x_{\lambda'} \in U_{\lambda'} \subseteq U$, meaning the net $\mathbf{x}$ converges to x.

($\Longleftarrow$): Suppose $\mathbf{x} = (x_\lambda)_{\lambda \in \Lambda}$ is a net of elements in A that converges to $x \in X$. For any open neighborhood U of x, note there is some $\lambda \in \Lambda$ such that $x_\lambda \in U$. Since $x_\lambda \in A$, U intersects A, meaning $x \in \overline{A}$. ❦

[3.8.9] Theorem (*Nets Probe Continuity*)

Let $f : X \to Y$. Then f is continuous if and only if, for every convergent net $\mathbf{x} = (x_\lambda)_{\lambda \in \Lambda} \to x$, the net $f(x_\lambda) \to f(x)$.

Proof.

($\Longrightarrow$): Suppose f is continuous. Let V be any open neighborhood of $f(x)$. By continuity, note that $U = f^{-1}(V)$ is an open neighborhood of x such that $f(U) \subseteq V$. Then there is some $\lambda' \in \Lambda$ such that $x_\lambda \in U$ for each $\lambda \geq \lambda'$. Thus $f(x_\lambda) \in f(U) \subseteq V$ for every $\lambda \geq \lambda'$, meaning $f(x_\lambda) \to f(x)$.

($\Longleftarrow$): Let A be any subset of X. For any $x \in \overline{A}$, note there is some convergent net $\mathbf{x} = (x_\lambda)_{\lambda \in \Lambda}$ drawing values from A such that $\mathbf{x} \to x$. By hypothesis, note $f(\mathbf{x}) \to f(x) \in \overline{f(A)}$. Thus $f(\overline{A}) \subseteq \overline{f(A)}$, completing the proof. ❦

Nets seem nice so far, but the true problem is that subsequences don't generalize well. If we generalized subsequences naively by just considering a subset of indices, subnets would be too weak to do anything.

[3.8.10] Remark (*The Failure of Naive Subnets*)

Suppose we tried to define a subnet by restricting f to a cofinal subset $J' \subseteq J$, in exact analogy with passing to a subsequence. Taking $J = \mathbb{Z}_+$, this recovers precisely the notion of a subsequence, so any theorem we prove about such "subnets" would specialize to a theorem about subsequences. But we already know that subsequences fail to detect compactness: the space $[0,1]^{[0,1]}$ is compact by Tychonoff, and yet it contains a sequence with no convergent subsequence. Under the naive definition, then, this compact space would admit a net with no convergent subnet, and the characterization of compactness we are after would be false.

The obstruction is one of size. To converge in a product, a net must eventually settle in every coordinate at once, and here there are uncountably many coordinates to satisfy simultaneously. A cofinal subset of $\mathbb{Z}_+$ is still countable, so no matter how cleverly we thin the indices, we have only countably many choices with which to meet uncountably many demands. Thinning cannot help us.

The repair, then, is not to select *fewer* indices but to permit *more*. By allowing an entirely new directed set K together with a map $g : K \to J$, we free ourselves from the size of J altogether: K may be far larger than J, and g need not be injective, so the subnet is permitted to revisit indices and to be indexed by a set of much greater cardinality than the original net. What remains is to insist that K still traverses J faithfully, and this is exactly what monotonicity and cofinality accomplish.

[3.8.11] Definition (*Subnet*)

Let $f : J \to X$ be a net relative to some directed set J. If K is a directed set together with some map $g : K \to J$ such that

(1) For any $i, j \in K$ such that $i \leq j$, we have $g(i) \leq g(j)$ (monotonicity),

(2) $g(K)$ is cofinal in J,

then we say the map $f \circ g : K \to X$ is a **subnet** of f.

These properties allow the subnet $f \circ g$ relative to K to lift properties of subsequences to subnets: monotonicity ensures we do not jump backward in the net, and cofinality ensures the subnet marches arbitrarily far out into the original directed set J.

[3.8.12] Theorem (*Convergence of Subnets*)

If a net converges to a point, then all subnets must converge to the same point.

[3.8.13] Definition (*Accumulation Point*)

Let $(x_\lambda)_{\lambda \in \Lambda}$ be a net in X. We say that x is an **accumulation point** of the net if, for any open neighborhood U of x, the set of indices λ such that $x_\lambda \in U$ is cofinal in Λ.

[3.8.14] Remark (*Cofinality and Accumulation Points*)

For sequences, an accumulation point is one that the sequence visits "infinitely often." Cofinality is the exact topological translation of "infinitely often" for nets. If x is an accumulation point, then no matter how far out we march into the directed set Λ, we can always find an index further out where the net jumps back into U. Because this set of "good" indices is cofinal, we can build a new directed set K that maps entirely into them, creating a subnet that stays inside U permanently (i.e., a subnet that converges to x).

[3.8.15] Theorem (*Accumulation Points are Limit Points of Subnet*)

A net has an accumulation point if and only if a subnet converges to the accumulation point.

[3.8.16] Theorem (*Equivalence of Compactness and Sequential Compactness via Nets*)

A space X is compact if and only if every net in X has a convergent subnet.

The aforementioned theorem says that compact spaces are sequentially compact relative to nets (i.e., if every net has an accumulation point). This closely mirrors the equivalence of compactness and sequential compactness in metric spaces.