7 Complete Metric Spaces and Function Spaces

7.1 Complete Metric Spaces

Before we proceed, we will revisit standard metrics and properties about them.

[7.1.1] Recap

(1) $\mathbb{R}^n$ has two common metrics: the **Euclidean metric**

$$d(x,y) = \left[\sum_{i=1}^{n}(x_i - y_i)^2\right]^{1/2}$$

and the **square metric** $\rho(x,y) = \max_{1\le i\le n}|x_i - y_i|$.

(2) By simple geometry, note that $\rho(x,y) \le d(x,y) \le \sqrt{n}\rho(x,y)$ for $x,y \in \mathbb{R}^n$. Thus properties of ρ (such as sequence convergence, sequences being Cauchy, the topology generated by the metric) are brought to d immediately by this inequality. Note that the constant $\sqrt{n}$ is the only obstruction, and it is exactly what fails in infinite dimensions (since $\sqrt{n}\to\infty$ as $n\to\infty$).

(3) It is easy to show the metric topology generated by ρ coincides perfectly with the product topology on $\mathbb{R}^n$. By the inequality from before, the metric topology generated by d also coincides with the product topology on $\mathbb{R}^n$.

(4) Note that neither metric generalizes cleanly to infinite products, since both may take the value ∞. Instead, given *any* metric d on a space X, define the **standard bounded metric**

$$\overline{d}(x,y) = \min\{d(x,y),1\}.$$

This induces the same topology as d, so boundedness is not a topological property; it depends on the choice of metric. Truncating in this way costs nothing topologically and is what makes the infinite constructions below well-defined.

(5) We now generalize the square metric to $\mathbb{R}^\omega$. The most direct approach is the **uniform metric**

$$\overline{\rho}(x,y) = \sup_{n\in\mathbb{Z}_+}\{\overline{d}(x_n,y_n)\},$$

replacing max with sup and truncating the distances above by 1. The topology this induces is the **uniform topology**. Alternatively, we may define the **product metric**

$$D(x,y) = \sup_{n\in\mathbb{Z}_+}\{\overline{d}(x_n,y_n)/n\},$$

which is the same as the uniform metric, but damps the nth coordinate so that later coordinates are less "important." Fittingly, it is the metric that produces the product topology on $\mathbb{R}^\omega$.

(6) These metrics can be placed on the space Y^X, the set of functions $f : X \to Y$. Then the product metric sets the mode of convergence to pointwise on this function space, whereas the uniform metric sets the mode to uniform convergence.

(7) In defining metrics on function spaces, consider the space $B(X,Y)$ of bounded functions $f : X \to Y$. We may define the **sup metric**

$$\rho(f,g) = \sup_{x\in X}\{d(f(x),g(x))\}.$$

Note that we do not need to use the bounded metric $\overline{d}$, since the functions are already bounded above, and so sup works out of the box. In general, note that the relationship between the uniform and sup metrics are simply $\overline{\rho}(f,g) = \min\{\rho(f,g),1\}$.

(8) Note that the product topology is strictly coarser than the uniform topology, which is strictly coarser than the box topology.

[7.1.2] Definition (*Cauchy Sequence, Complete Metric Space*)

Let (X,d) be a metric space. A sequence $(x_n) \subseteq X$ is said to be **Cauchy** if, for any $\varepsilon > 0$, there is some $N \in \mathbb{Z}_+$ such that $d(x_m, x_n) < \varepsilon$ for every $m, n \geq N$. We say the metric space X is **complete** if every Cauchy sequence converges.

[7.1.3] Theorem (*Equivalence of Convergent Sequences and Cauchy Sequences in Metric Spaces*)

In a complete metric space (X,d), a sequence converges if and only if it is Cauchy.

This fact is important in analysis since, in complete metric spaces, we can demonstrate a sequence converges if it is Cauchy, without needing a candidate for the limit.

[7.1.4] Theorem (*Closed Subspace of Complete Metric Space is Complete*)

A closed subspace of a complete metric space is complete.

Proof. Of course, a subspace topology derived from the metric topology is metric. Consider any Cauchy sequence contained in the closed subspace. It necessarily converges to a limit point of the closed subspace, but closed spaces contain all their limit points, and so the Cauchy sequence converges to a point in the closed subspace. Thus the closed subspace is complete. ❦

[7.1.5] Theorem (*Complete if Every Cauchy Sequence has Accumulation Point*)

A metric space (X,d) is complete if every Cauchy sequence has a convergent subsequence.

Proof. Suppose if $(x_n) \subseteq X$ is a Cauchy sequence with a subsequence $x_{n_k} \to x$. For any $\varepsilon > 0$, note there is some $N_1 \in \mathbb{Z}_+$ such that $d(x_n, x_{n_k}) < \varepsilon/2$ for every $n, n_k \geq N_1$, since the sequence is Cauchy. Moreover, there is some $N_2 \in \mathbb{Z}_+$ such that $d(x_{n_k}, x) < \varepsilon/2$ for every $n_k \geq N_2$ by definition of convergence. Thus, for any $n \geq \max\{N_1, N_2\}$, we have that

$$d(x_n, x) \leq d(x_n, x_{n_k}) + d(x_{n_k}, x) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Since all Cauchy sequences are convergent, the space is complete. ❦

[7.1.6] Theorem (*Euclidean Space is Complete in its Standard Metrics*)

Euclidean space is complete under the standard and square metrics.

Proof. Let $(\mathbb{R}^n, \rho)$ be Euclidean space endowed with the square metric. For any Cauchy sequence x_n in this metric space, there is some $N \in \mathbb{Z}_+$ such that $\rho(x_m, x_n) < 1$ for every $m, n \geq N$. Then choose a number

$$M = \max\{\rho(x_0, \mathbf{0}_n), \rho(x_1, \mathbf{0}_n), \ldots, \rho(x_{N-1}, \mathbf{0}_n), \rho(x_N, \mathbf{0}_n) + 1\}.$$

Note then that x_n is fully contained in $[-M, M]^n$. Note this box is compact, and so x_n has a convergent subsequence by sequential compactness. Thus Cauchy sequences converge and so the space is complete.

Finally, for $(\mathbb{R}^n, d)$, note that a sequence converges if and only if it converges with respect to ρ, and the same for the sequence being Cauchy. ❦

[7.1.7] Theorem (*Sequence in Product Space Converges iff. Components Converge*)

Let $\{X_\lambda\}_{\lambda\in\Lambda}$ be an arbitrary collection of spaces with $X = \prod_{\lambda\in\Lambda} X_\lambda$ its product. Then a sequence $\mathbf{x}_n \to \mathbf{x}$ in the product space if and only if $\pi_\lambda(\mathbf{x}_n) \to \pi_\lambda(\mathbf{x})$ for every $\lambda \in \Lambda$.

Proof.

($\Longrightarrow$): Note that π_λ is continuous for every $\lambda \in \Lambda$, so the image of $\mathbf{x}_n$ under each projection maps to the projection of $\mathbf{x}$.

($\Longleftarrow$): Let $\mathbf{x}_n$ be some sequence such that there exists some $\mathbf{x} \in X$ for which $\pi_\lambda(\mathbf{x}_n) \to \pi_\lambda(\mathbf{x})$ for every $\lambda \in \Lambda$. Let U be any basic open neighborhood of $\mathbf{x}$. By definition, we have that there exist indices $\{\lambda_k\}_{k=1}^n$ such that

$$U = \bigcap_{k=1}^{n} \pi_\lambda^{-1}(U_\lambda),$$

where $U_{\lambda_k} \subseteq X_{\lambda_k}$ is open. For each $k \in \{1, \ldots, n\}$, there is some $N_k \in \mathbb{Z}_+$ such that $\pi_{\lambda_k}(\mathbf{x}_n) \in \pi_{\lambda_k}(U_\lambda)$ for every $n \geq N_k$. Choosing $N = \max\{N_k\}_{k=1}^n$, note that $\mathbf{x}_n \in U$ for every $n \geq N$, completing the proof. ❦

[7.1.8] Theorem (*$(\mathbb{R}^\omega, D)$ is complete*)

The metric space $(\mathbb{R}^\omega, D)$, with $D(x,y) = \sup_{n\in\mathbb{Z}_+}\{\overline{d}(x,y)/n\}$ is complete.

Proof. Suppose $\mathbf{x}_n$ is Cauchy in this metric space. Note that

$$\overline{d}(\pi_n(\mathbf{x}_n), \pi_n(\mathbf{y}_n)) \leq nD(\mathbf{x}, \mathbf{y}),$$

and so $\pi_n(\mathbf{x}_n)$ is a Cauchy sequence in $(\mathbb{R}, \overline{d})$ for each $n \in \mathbb{Z}_+$. This metric space is complete, and so the projections of the sequence converge. By the previous theorem, this means $\mathbf{x}_n$ converges. ❦

[7.1.9] Example (*Completeness is not a Topological Property*)

Note that $\mathbb{R}$ is complete in its usual metric. Now consider the subspace $(-1,1)$ with the same metric, $d(x,y) = |x-y|$. But note the Cauchy sequence $x_n = 1 - 1/n$ converges to 1, which is outside of this interval. Thus this space is not complete, even though $(-1,1) \cong \mathbb{R}$. Thus completeness is not a topological property.

[7.1.10] Definition (*Uniform Metric, Revisited*)

Let (Y,d) be a metric space, together with an induced bounded metric $\overline{d}(x,y) = \min\{d(x,y), 1\}$. Let Y^X be the space of functions $\{f \mid f : X \to Y\}$. Then we define the **uniform metric**

$$\overline{\rho}(f,g) = \sup_{x\in X}\{d(f(x), g(x))\}.$$

[7.1.11] Remark

Note that if X is an indexing set, the space Y^X is the set of sequences in Y indexed by X. Then the uniform metric defined for sequences coincides with the uniform metric defined in terms of functions. Of course, this is all true because a sequence is truly a function from the indexing set into Y.

[7.1.12] Theorem (*Criterion for Complete Metric Space Under Unifom Metric*)

Suppose (Y, d) is a complete metric space. Then the metric space $(Y^X, \overline{\rho})$ is complete as well.

Proof. Let $(f_n)_{n\in\mathbb{Z}_+}$ be a Cauchy sequence in Y^X, where each $f_n : X \to Y$. For a fixed $x \in X$, consider the sequence $(f_n(x))_{n\in\mathbb{Z}_+}$ in Y. Fix $1 \geq \varepsilon > 0$, and let $N \in \mathbb{Z}_+$ such that $\overline{\rho}(f_m, f_n) < \varepsilon$ for every $m, n \geq N$. Then, for any $m, n \geq N$, note that

$$d(f_m(x), f_n(x)) < \sup_{x\in X}\{d(f_m(x), f_n(x))\} = \overline{\rho}(f_m, f_n) < \varepsilon,$$

and so $(f_n(x))$ is Cauchy. Since Y is complete, we know this sequence converges. Thus, for every $x \in X$, define $f(x)$ as the value $(f_n(x))$ converges to.

Finally, we show $f_n \to f$ in Y^X. Fix some $\varepsilon > 0$, then note there is some $N \in \mathbb{Z}_+$ such that $\overline{\rho}(f_m, f_n) < \varepsilon$ for every $m, n \geq N$. Fix n and let $m \to \infty$, then note that by the limit comparison theorem and continuity of the metric, $\overline{\rho}(f, f_n) \leq \varepsilon$. Thus $f_n \to f$.

If ε were chosen to be greater than 1 in either case, the bounded metric $\overline{\rho}$ would evaluate to 1, making the proceeding statements trivial. ❦

Now, when defining Y^X for Y a metric space, let X be a topological space. The set Y^X is not different, but we may now consider subsets that we were unable to do before.

[7.1.13] Definition (*Space of Continuous and Bounded Functions*)

Let X be a topological space and Y be a metric space. The **space of continuous maps**, denoted $C(X, Y) \subseteq Y^X$, is given by

$$C(X, Y) = \{f \mid f : X \to Y \text{ is continuous}\}.$$

Similarly, the **space of bounded maps**, denoted $B(X, Y) \subseteq Y^X$, is given by

$$B(X, Y) = \{f \mid f : X \to Y \text{ is bounded}\}.$$

Recall that a map $f : X \to Y$, with Y a metric space, is said to be **bounded** if $f(X)$ is contained in a single ball.

[7.1.14] Theorem

Let X be a topological space and (Y, d) a metric space. Then the spaces $C(X, Y)$ and $B(X, Y)$ are closed.

Proof. Note these spaces are metric, so sequences probe limit points. We show that any sequences contained in these spaces converge inside the space.

Let (f_n) be a sequence of functions in $C(X, Y)$, meaning each $f_n : X \to Y$ is continuous, such that $f_n \to f$. Note that convergence in this space is uniform convergence, so f is automatically continuous and so $f \in C(X, Y)$. Since $f_n \to f$, for any $\varepsilon > 0$ there is some $N \in \mathbb{Z}_+$ such that $\overline{\rho}(f_n, f) < \varepsilon$. Then, for any $x \in X$, note that

$$\overline{d}(f_n(x), f(x)) < \overline{\rho}(f_n, f) < \varepsilon.$$

Note that ε is an upper bound for $\overline{d}(f_n(x), f(x))$ for *every* $x \in X$, independent of ε. Thus $f_n \to f$ uniformly, and so f is continuous, meaning $f \in C(X, Y)$ and so the space is closed.

Now let (f_n) be a sequence of bounded functions in $B(X, Y)$, meaning there is some $\varepsilon_n > 0$ for each $f_n : X \to Y$ such that $f_n(X) \subseteq B_{\varepsilon_n}(\mathbf{0})$. Note that for $\varepsilon = 1$, there is some $N \in \mathbb{Z}_+$ such that $d(f_n, f) < 1$ for every $n \geq N$. Note then that

$$d(\mathbf{0}, f) \leq d(\mathbf{0}, f_n) + d(f_n, f) \leq \varepsilon_n + 1$$

for every $n \geq N$. Thus $f \in B_{\varepsilon_N+1}(\mathbf{0})$, meaning $f \in B(X, Y)$. Thus the space is closed, completing the proof. ❦

[7.1.15] Theorem

Let X be a topological space and (Y,d) a *complete* metric space. Then the spaces $C(X,Y)$ and $B(X,Y)$ are *complete*.

[7.1.16] Definition (*Sup Metric*)

If (Y,d) is a metric space, one can define another metric, the **sup metric**, on the set $B(X,Y)$ of bounded functions $X \to Y$ by the equation

$$d_\infty(f,g) = \sup_{x \in X}\{d(f(x),g(x))\}.$$

Note that this metric is well defined since $f(X) \cup g(X)$ is bounded above, and so $d(f(x),g(x))$ is bounded above for all $x \in X$. In fact, the sup metric may be put on all other bounded sets. The sup metric is almost identical to the uniform metric, in which $\overline{\rho}(f,g) = \min\{d_\infty(f,g),1\}$.

[7.1.17] Theorem (*Metric Space Completion Theorem*)

Let (X,d) be a metric space. Then there is an isometric embedding of X into a complete metric space.

Proof. Let $x_0 \in X$ be fixed. For any $a \in X$, define the map

$$\varphi_a(x) = d(x,a) - d(x,x_0).$$

We show $\varphi_a \in B(X,\mathbb{R})$. Indeed, note that

$$\begin{aligned} d(x,a) &\le d(x,x_0) + d(x_0,a), \\ d(x,x_0) &\le d(x,a) + d(a,x_0). \end{aligned}$$

Then we have that

$$\Big(|\varphi_a(x)| = |d(x,a) - d(x,x_0)|\Big) \le |d(x,x_0) - d(x,a)| \le d(x,x_0) + d(x,a) \le d(x_0,a).$$

Thus $|\varphi_a(x)| \le |d(x_0,a)|$ for every $x \in X$, meaning φ_a is bounded and so $\varphi_a \in B(X,\mathbb{R})$. Finally, define the map

$$\Phi_a : X \to B(X,\mathbb{R}) \quad \Phi_a(x) = \varphi_a(x).$$

Note that $B(X,\mathbb{R})$, where $\mathbb{R}$ takes the Euclidean metric, is complete with the sup metric. We show Φ_a is an isometry. Note that an isometry between metric spaces automatically constitutes an embedding, completing the proof.

To show Φ_a is an isometry, we show

$$\rho(\varphi_a,\varphi_b) = d(a,b).$$

For any $a,b \in X$, note that

$$\begin{aligned} \rho(\varphi_a,\varphi_b) &= \sup_{x \in X}\Big\{d(\varphi_a(x),\varphi_b(x))\Big\} \\ &= \sup_{x \in X}\Big\{\big|\varphi_a(x) - \varphi_b(x)\big|\Big\} \\ &= \sup_{x \in X}\Big\{|(d(x,a) - d(x,x_0)) - (d(x,b) - d(x,x_0))|\Big\} \\ &= \sup_{x \in X}\Big\{|d(x,a) - d(x,b)|\Big\} \\ &\le d(x,a) + d(x,b) \le d(a,b). \end{aligned}$$

Finally, note that this inequality cannot be strict. Take $x = a$, and note

$$|d(x,a) - d(x,b)| = |d(a,b)| \not< d(a,b).$$

[7.1.18] Definition (*Completion of a Metric Space*)

Let (X,d) be any metric space. Let $h: X \to Y$ be an isometric embedding into a complete metric space. Then the space $\overline{h(X)}$ is a complete metric space in Y, which we call the **completion**.

[7.1.19] Example (*Completion of* $(\mathbb{Q},d)$)

A classic result is that the completion of $\mathbb{Q}$ under the metric $d(x,y) = |x-y|$ is $\mathbb{R}$ equipped with the standard Euclidean metric. Using the isometric embedding Φ into the complete space $B(\mathbb{Q},\mathbb{R})$, we map any $a \in \mathbb{Q}$ to the function $\varphi_a \in B(\mathbb{Q},\mathbb{R})$ defined by $\varphi_a(x) = |x-a| - |x|$ (where $x \in \mathbb{Q}$).

Consider, for example, a Cauchy sequence $(a_n) \subseteq \mathbb{Q}$ that approximates $\sqrt{2}$. In $\mathbb{Q}$, this sequence has no limit. However, the sequence of their images $\Phi(a_n) = \varphi_{a_n}$ is Cauchy in $B(\mathbb{Q},\mathbb{R})$. Because $B(\mathbb{Q},\mathbb{R})$ is complete, this sequence of functions must converge uniformly. Indeed, $\varphi_{a_n}(x) = |x-a_n| - |x|$ converges uniformly to the function $f(x) = |x-\sqrt{2}| - |x|$. This limit function f is a well-defined bounded function in $B(\mathbb{Q},\mathbb{R})$, but it is not in the image $\Phi(\mathbb{Q})$. The "hole" at $\sqrt{2}$ has thus materialized as a concrete boundary point of $\Phi(\mathbb{Q})$ in the function space.

Repeating this process for every Cauchy sequence in $\mathbb{Q}$ fills in all such holes. The closure of the image of $\mathbb{Q}$ is therefore exactly:

$$\overline{\Phi(\mathbb{Q})} = \Big\{ f_r(x) = |x-r| - |x| : r \in \mathbb{R} \Big\}.$$

This closed subspace of $B(\mathbb{Q},\mathbb{R})$ is, by definition, the completion of $\mathbb{Q}$. Note that this abstract space of functions is structurally identical to $\mathbb{R}$. The image of $\mathbb{R}$ under this isometry is precisely $\overline{\Phi(\mathbb{Q})}$, and so there is a surjective isometry between them. Since a surjective isometry is a perfect equivalence of metric spaces, we safely identify the completion of $\mathbb{Q}$ directly with $\mathbb{R}$.

[7.1.20] Remark (*Intuition behind Completions*)

A completion effectively fills the holes in a metric space caused by Cauchy sequences that fail to converge. A Cauchy sequence consists of points that get arbitrarily close to one another; geometrically, it *ought* to converge to a limit. When a space is incomplete, it simply lacks the points that these sequences are aiming at. The completion process adjoins exactly these missing limit points. By embedding X isometrically into the humongous *complete* function space $B(X,\mathbb{R})$, we force the missing limit functions to exist. Thus we patch the holes of X and guarantee that every Cauchy sequence in the new space converges.

7.1.1 Exercises

[7.1.21] Problem

Let (X,d) be a metric space. Suppose that, for some $\varepsilon > 0$, every ε-ball in X has compact closure. Show that X is complete.

Proof. Let (x_n) be any Cauchy sequence. There exists some $N \in \mathbb{Z}_+$ such that $d(x_m, x_n) < \varepsilon/2$ for every $m, n \geq N$. Then there exists finitely many closed ε-balls whose union contains $x_1, \ldots, x_{N-1}$. Adjoin to this list of balls a single ball of radius ε containing all $x_N, x_{N+1}, \ldots$. Let this list be $\{U_1, \ldots, U_n\}$, and note that $\{\overline{U_1}, \ldots, \overline{U_n}\}$ still covers the sequence. Since each closure of the finitely many sets is compact, the set $U = \bigcup_{k=1}^{n} U_k$ is compact. Since x_n is contained in the compact set U, and compactness is equivalent to sequential compactness in metrizable spaces, we have that x_n admits a convergent subsequence. Thus x_n is a convergent sequence, making X complete. ❦

[7.1.22] Problem

Let (X, d_X) and (Y, d_Y) be metric spaces, with Y complete. Let $A \subseteq X$. Show that, for any uniformly continuous $f : A \to Y$, there exists a *unique*, uniformly continuous extension to $g : \overline{A} \to Y$.

Proof. Let $x \in \overline{A} \setminus A$. Then there exists a sequence (x_n) in A such that $x_n \to x$. By uniform continuity of f, for any $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$d_X(x_1, x_2) < \delta \implies d_Y(f(x_1), f(x_2)) < \varepsilon$$

for every $x_1, x_2 \in A$. There is some $N \in \mathbb{Z}_+$ such that $d_X(x_m, x_n) < \delta$ for every $m, n \geq N$, and so $d_Y(f(x_m), f(x_n)) < \varepsilon$ for every $m, n \geq N$. Since ε is arbitrary, we have that $f(x_n)$ is Cauchy in the complete space Y. Thus $f(x_n) \to y_x$.

Now suppose (z_n) is some other sequence in A such that $z_n \to x$. Fix $\varepsilon > 0$. Once again, by uniform continuity, there exists a $\delta > 0$ such that

$$d_X(z_n, x_n) < \delta \implies d_Y(f(z_n), f(x_n)) < \varepsilon$$

for every $n \in \mathbb{Z}_+$. Since $z_n \to x$ and $x_n \to x$, note that $d_X(z_n, x_n) \to 0$, and so $d_Y(f(z_n), f(x_n)) \to 0$. Since $f(x_n) \to y_x$, we have that $f(z_n) \to y_x$.

Define a function $g : \overline{A} \to Y$ such that $g|_A = f$ and $g|_{\overline{A} \setminus A}(x) = y_x$, where y_x is chosen as per before. Note that any sequence tending to the same point x in $\overline{A} \setminus A$ has the same limit point y_x in the image, and so g is well-defined. We prove this function is uniformly continuous.

Fix $\varepsilon > 0$. By the uniform continuity of $g|_A = f$, there exists $\delta > 0$ such that

$$d_X(x_1, x_2) < \delta \implies d_Y(g(x_1), g(x_2)) < \varepsilon$$

for every $x_1, x_2 \in A$. We show this δ works for all points in $\overline{A}$. Indeed, suppose $u, v \in \overline{A}$ are such that $d_X(u, v) < \delta$. There exist sequences (u_n) and (v_n) in A such that $u_n \to u$ and $v_n \to v$. Because the metric d_X is continuous, $\lim_{n \to \infty} d_X(u_n, v_n) = d_X(u, v) < \delta$. Thus, there exists some $N \in \mathbb{Z}_+$ such that $d_X(u_n, v_n) < \delta$ for every $n \geq N$.

Applying the uniform continuity of $g|_A$ to these sequence elements, we find that $d_Y(g(u_n), g(v_n)) < \varepsilon$ for every $n \geq N$. By the construction of g, we know $g(u_n) \to g(u)$ and $g(v_n) \to g(v)$. Taking the limit as $n \to \infty$ and using the continuity of the metric d_Y, we obtain $d_Y(g(u), g(v)) \leq \varepsilon$. Since $\varepsilon > 0$ was arbitrary, we conclude that g is uniformly continuous on $\overline{A}$.

Recall that continuous functions preserve limits. g does this, since for every sequence $x_n \to x$, we define $g(x)$ such that $g(x_n) \to g(x)$. If $g|_{\overline{A} \setminus A}$ were defined in any other way, the resulting function would not be continuous. Thus g is the unique, uniformly continuous extension of f.

[7.1.23] Problem

Let (X, d) be a metric space. Recall that a map $f : X \to X$ is said to be a **contraction map** if there exists some $C < 1$ such that

$$d(f(x), f(y)) \leq C d(x, y).$$

Show that if X is complete, then any contraction map admits a *unique* fixed point.

Proof. Fix any $x_0 \in X$, then consider the sequence x_n defined by $x_n = f^{n-1}(x_0)$, where $f^0 = \mathrm{id}$.

We will show this sequence is Cauchy. Fix $\varepsilon > 0$. Note that $d(x_n, x_{n+1}) \leq C d(x_{n-1}, x_n)$ for any $n \in \mathbb{Z}_+$. Then note

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + \cdots + d(x_{n-1}, x_n) \\ &\leq C^m \underbrace{d(x_0, x_1)}_{:=r} + C^{m+1} d(x_0, x_1) + \cdots + C^{n-1} d(x_0, x_1) \\ &= r C^m \left(\frac{1 - C^{n-m}}{1 - C} \right) = r \left(\frac{C^m - C^n}{1 - C} \right). \end{aligned}$$

Note that $d(x_m, x_n)$ can be made arbitrarily small since $C < 1$ (and so $C^m - C^n \to 0$ as $m, n \to \infty$). Thus there is some $N \in \mathbb{Z}_+$ such that $d(x_m, x_n) < \varepsilon$ for every $m, n \geq N$. Thus the sequence is Cauchy and thus convergent, and so $x_n \to x^*$. We show x^* is our desired fixed point.

Note that f is C-Lipschitz and thus continuous, so $f(x_n) \to f(x^*)$. Of course, $d(f(x_n), x_n) \to 0$, and so they converge to the same limit. Thus $x^* = f(x^*)$.

Finally, we show x^* is the unique fixed point. Suppose $x' \in X$ is another fixed point. Then note $d(f(x^*), f(x')) = d(x^*, x') \leq Cd(x^*, x')$. Then either $C = 0$, in which there is only one unique fixed point by inspection, or $d(x^*, x') = 0$ (and so $x^* = x'$), completing the proof.

7.2 Compactness in Metric Spaces

We have already shown a plethora of equivalences of compactness in metric spaces. Namely, being compact, sequentially compact, limit point compact, and having the finite intersection property are all equivalent in metric spaces.

Immediately, we note that compact metric spaces are complete. Indeed, compactness is equivalent with sequential compactness and so Cauchy sequences have convergent subsequences, so the space is complete. Complete metric spaces are not automatically compact, however (consider $\mathbb{R}$). We seek to find a condition on metric spaces that, together with completeness, is equivalent to compactness.

[7.2.1] Definition (*Total Boundedness*)

A metric space (X, d) is said to be **totally bounded** if, for any $\varepsilon > 0$, there exist finitely many ε-balls that cover X.

[7.2.2] Remark (*Total Boundedness and Boundedness*)

Note that any totally bounded space is bounded. Indeed, choose $\varepsilon = 1/2$ such that $\{B(x_k, 1/2)\}_{k=1}^n$ covers the entire space. From this, we have that $1 + \max\{d(x_i, x_j)\}_{i,j=0}^n$ is an upper bound on the diameter of the space, implying it is bounded. Bounded spaces are not totally bounded, though. $\mathbb{R}$ is bounded under the standard bounded metric $\overline{d}(x, y) = \min\{1, |x - y|\}$, but is not totally bounded (a countably infinite number of balls of radius ≤ 1 would be needed to cover $\mathbb{R}$).

Take the metric $d(x, y) = |x - y|$ and the space $\mathbb{R}$. It is complete but not totally bounded. Now consider $(-1, 1)$. It is totally bounded but not complete. Finally, note that $[-1, 1]$ is both complete and totally bounded.

The example from before illustrates an example of a crucial theorem. A metric space is compact if and only if it is complete *and* totally bounded. Note this resembles the theorem in $\mathbb{R}$ that a set is compact iff closed (i.e., complete) and bounded (i.e., totally bounded).

[7.2.3] Theorem (*Heine-Borel for Metric Spaces*)

A metric space is compact if and only if it is complete and totally bounded.

Proof.

($\Longrightarrow$): Suppose X is compact. Then X is sequentially compact, meaning any Cauchy sequence admits a convergent subsequence, making X complete. Moreover, for any $\varepsilon > 0$, the open cover $\{B(x, \varepsilon)\}_{x \in X}$ has a finite subcover $\{B(x_k, \varepsilon)\}_{k=1}^n$, making X totally bounded.

($\Longleftarrow$): Suppose X is complete and totally bounded. Let $(x_n) \subseteq X$ be any sequence. Since X is totally bounded, there exists a finite set of balls $\{B(x^k, 1)\}_{k=1}^n$ such that $x_n \in \bigcup_{k=1}^n B(x^k, 1)$ for every $n \in \mathbb{Z}_+$. By pigeonhole, there exists some $j \in \{1, \ldots, n\}$ such that infinitely many points of the sequence lie in $B(x^j, 1)$. Choose a single point in the ball from this sequence and label it x_{n_1}. Now note we may partition the ball $B(x^j, 1)$ by finitely many balls of radius $1/2$, find infinitely many points of x_n inside one of these balls, and label it x_{n_2} (such that $n_2 > n_1$). Repeat this for ball of radius $1/4$ and choosing x_{n_3} (such that $n_3 > n_2 > n_1$), and by induction, choose x_{n_k} through the ball of radius $1/2^{k-1}$ (such that n_k is greater than all previous choices). Since $1/2^{k-1} \to 0$, we have that there exists some $N \in \mathbb{Z}_+$ such that $d(x_{n_j}, x_{n_k}) < \varepsilon$ for every $j, k \geq N$. Thus any sequence x_n has a Cauchy subsequence, and since X is complete, it is convergent. Thus X is sequentially compact and thus compact.

The goal for the rest of this section is to characterize the compact subsets of $(C(X,\mathbb{R}^n),\rho)$, the metric space of continuous maps from a topological space X to $\mathbb{R}^n$ together with the sup metric. We use the result from before to identify compact subsets, culminating into what is known as the Arzela-Ascoli theorem. This theorem characterizes all compact subsets of $C(X,\mathbb{R}^n)$ with the sup metric as precisely the subsets that are closed, uniformly bounded, and equicontinuous.

[7.2.4] Definition (*Equicontinuity*)

Suppose X is a topological space and (Y,d) is a metric space. Let $\mathcal{F}$ be a family of functions $X \to Y$. For any $x_0 \in X$, we say the family $\mathcal{F}$ is **equicontinuous at** x_0 if, for any $\varepsilon > 0$, there exists a single neighborhood $U \subseteq X$ of x_0 such that, for every $x \in X$ and $f \in \mathcal{F}$,

$$d(f(x), f(x_0)) < \varepsilon.$$

If $\mathcal{F}$ is equicontinuous at all points in X, we say the family is simply equicontinuous.

Recall that for f to be continuous at x_0, we need that for any $\varepsilon > 0$ there exists an open neighborhood U of x_0 such that

$$d(f(x), f(x_0)) < \varepsilon$$

for every $x \in U$. Compare this to equicontinuity: for $\mathcal{F}$ to be equicontinuous at x_0, we need this relation to hold for every $f \in \mathcal{F}$ for only one U. Note that if X is metric, then the open neighborhood U of x_0 would just be a δ-neighborhood.

[7.2.5] Theorem

Suppose X is a topological space and (Y,d) a metric space. If $\mathcal{F} \subseteq C(X,Y)$ is totally bounded under the uniform metric $\overline{\rho}$, then it is equicontinuous under the metric d.

Proof. Suppose $\mathcal{F}$ is totally bounded. Fix $x_0 \in X$ and $\varepsilon \in (0,1)$. We seek to show there exists a neighborhood U of x_0 such that $d(f(x), f(x_0)) < \varepsilon$ for every $x \in U$ and $f \in \mathcal{F}$. Since $\varepsilon < 1$, we use d and $\overline{d}$ interchangeably, for they are equivalent.

Let $\delta = \varepsilon/3$. Since $\mathcal{F}$ is totally bounded, note there exists finitely many open balls $\{B_{\overline{\rho}}(f_k,\delta)\}_{k=1}^n$ from $C(X,Y)$ that covers $\mathcal{F}$. Note that each f_k is continuous, and so we can choose an open neighborhood U of x_0, we have that

$$d(f_k(x), f_k(x_0)) < \varepsilon$$

for every $x \in U$ and $k \in \{1,\dots,n\}$.

Fix any $f \in \mathcal{F}$. From the finite covering of $\mathcal{F}$, choose some f_k such that $\overline{\rho}(f,f_k) < \delta$. Finally, note

$$\begin{aligned} d(f(x), f(x_0)) &\le d(f(x), f_k(x)) + d(f_k(x), f_k(x_0)) + d(f_k(x_0), f(x_0)) \\ &\le 3\delta = 3\varepsilon/3 = \varepsilon. \end{aligned}$$

We get this relationship from the fact that $d(f(*), f_k(*)) < \overline{\rho}(f,f_k) < \delta$, as well as how we chose U. If $\varepsilon > 1$, then just apply the same procedure to some $\varepsilon' < 1 < \varepsilon$. Thus $\mathcal{F}$ is equicontinuous. ❦

[7.2.6] Theorem

Suppose X is a topological space and (Y,d) is a metric space. If $\mathcal{F} \subseteq C(X,Y)$ is equicontinuous under d, as well as X and Y being compact, we have that $\mathcal{F}$ is totally bounded under the uniform and sup metrics.

In sum, if $\mathcal{F} \subseteq C(X,Y)$ is totally bounded it is equicontinuous, and if it is equicontinuous (together with X,Y compact) it is totally bounded.

[7.2.7] Definition (*Pointwise Bounded*)

Suppose X is a topological space and (Y,d) is a metric space. A family of functions $\mathcal{F} \subseteq C(X,Y)$ is said to be **pointwise bounded** under d if, for each $a \in X$, the set $\{f(a) : f \in \mathcal{F}\}$ is bounded under d.

[7.2.8] Definition (*Uniformly Bounded*)

Suppose X is a topological space and (Y,d) is a metric space. A family of functions $\mathcal{F} \subseteq C(X,Y)$ is said to be **uniformly bounded** under d if there exists a point $y_0 \in Y$ and a real number $M > 0$ such that

$$d(f(x), y_0) \leq M$$

for every $x \in X$ and $f \in \mathcal{F}$. If Y is a normed space such as $\mathbb{R}^n$, this is equivalent to saying there is some $M > 0$ such that $\|f(x)\| \leq M$ for all $x \in X$ and $f \in \mathcal{F}$. Note that $\mathcal{F}$ is uniformly bounded under d if and only if $\mathcal{F}$ is bounded as a subset of $C(X,Y)$ under the sup metric ρ.

[7.2.9] Theorem (*Arzela-Ascoli Theorem*)

Let X be a compact space and $\mathbb{R}^n$ Euclidean space together with either the Euclidean or square metric. Give $C(X,\mathbb{R}^n)$ the corresponding sup metric. A subspace $\mathcal{F} \subseteq C(X,\mathbb{R}^n)$ has compact closure if and only if it is equicontinuous and pointwise bounded under $\mathbb{R}^n$'s metric.

[7.2.10] Theorem (*Arzela-Ascoli Theorem*)

Let X be a compact space and $\mathbb{R}^n$ Euclidean space together with either the Euclidean or square metric. Give $C(X,\mathbb{R}^n)$ the corresponding sup metric. A subspace $\mathcal{F} \subseteq C(X,\mathbb{R}^n)$ is compact if and only if it is closed, bounded under the sup metric, and equicontinuous under $\mathbb{R}^n$'s metric.

[7.2.11] Remark

The entirety of this section is dedicated to answering one question: what does a compact set look like in an infinite-dimensional function space? In finite dimensions ($\mathbb{R}^n$), Heine-Borel tells us that a set is compact if and only if it is closed and bounded. In infinite dimensions, this fails; a sequence of functions bounded between -1 and 1 can oscillate infinitely fast, never converging to anything.

To fix this, we look to metric spaces in general: a space is compact if and only if it is complete and totally bounded. We simply translate these two topological conditions into analytic conditions for functions. Because the uniform metric space $C(X,\mathbb{R}^n)$ is already complete, demanding that a subspace $\mathcal{F}$ be complete is exactly equivalent to demanding that $\mathcal{F}$ be **closed**. Thus our task boils down to finding an analytic condition on functions that is equivalent to total boundedness. Indeed, we proved lemmas that if a family of functions can be covered by finitely many ε-balls (total boundedness), it is forced to be **equicontinuous**, which prevents the functions from oscillating wildly. This is an equivalence (i.e., the converse is true) if the domain and codomain of these functions are compact.

We are not done yet, however. To characterize the compact subsets of $C(X,\mathbb{R}^n)$, we cannot directly use this equivalence between total boundedness and equicontinuity. Indeed, the codomain $\mathbb{R}^n$ is not compact. This is why **pointwise boundedness** is introduced.

Suppose $\mathcal{F}$ is pointwise bounded and equicontinuous and X is compact. We show these conditions force the family to be uniformly bounded. For each $a \in X$, equicontinuity gives an open neighborhood U_a of a such that $d(f(x), f(a)) < 1$ for every $x \in U_a$ and $f \in \mathcal{F}$. Since $\mathcal{F}$ is pointwise bounded, we have that $d(f(a), \mathbf{0}) \leq M_a$ for each $a \in X$. Then for any $x \in U_a$ and any $f \in \mathcal{F}$, note that

$$d(f(x), \mathbf{0}) \leq d(f(x), f(a)) + d(f(a), \mathbf{0}) < 1 + M_a.$$

Thus each $f \in \mathcal{F}$ is bounded in each U_a. Since $\{U_a\}_{a \in X}$ is an open cover for X and X is compact, there is a finite subcover $\{U_{a_1}, \ldots, U_{a_k}\}$. Simply choose the maximum of the upper bounds $(M_{a_j} + 1)$ from this finite collection, and this is what every function is bounded above by on all of X. Thus $\mathcal{F}$ is uniformly bounded.

This guarantees that the images of all functions in $\mathcal{F}$ lie inside a finite-radius closed ball in $\mathbb{R}^n$. By the standard Heine-Borel theorem for Euclidean space, this closed ball is compact. Thus, enforcing $\mathcal{F}$ to be pointwise bounded gives us an easy route to equate total boundedness and equicontinuity, since the images of these functions all lie in a compact codomain. Assembling the pieces: $\mathcal{F}$ is equicontinuous and pointwise bounded if and only if it is totally bounded, which together with the completeness of $C(X, \mathbb{R}^n)$ characterizes the subspaces with compact closure. This is the first version of the theorem.

The two versions of the Arzela-Ascoli theorem state the exact same mathematical fact. Note that in any metric space, a set A is compact if and only if it is closed and its closure $\overline{A}$ is compact.

The first version tells us exactly when the closure $\overline{\mathcal{F}}$ is compact (when $\mathcal{F}$ is equicontinuous and pointwise bounded). The second version simply adds the "closed" requirement to the checklist so that $\mathcal{F} = \overline{\mathcal{F}}$, meaning $\mathcal{F}$ itself is compact.

Furthermore, the first version requires the weaker condition of *pointwise* boundedness, while the second version requires boundedness under the sup metric, i.e. *uniform* boundedness. This is a free upgrade: if a family of functions is equicontinuous and pointwise bounded on a compact domain, it is automatically uniformly bounded, as shown above. The first version is often preferred in practice because checking pointwise bounds is algebraically much easier than proving uniform bounds.

7.3 Pointwise and Compact Convergence

[7.3.1] Definition (*Topology of Pointwise Convergence*)

Let X be a topological space. Given any $x \in X$ and open set U of the space Y, let

$$S(x, U) = \{f : f \in Y^X \text{ and } f(x) \in U\}.$$

The collection of all $S(x, U)$ forms a subbasis for a topology on Y^X called the **topology of pointwise convergence**.

Let's consider the basis elements for this topology, which are simply finite intersections of subbasis elements. For some $f : X \to Y$, a basis element about f would comprise functions g that are "close" to f at finitely many points. For example, let

$$B = \bigcap_{k=1}^{n} S(x_k, U_k)$$

be an arbitrary basis element about an arbitrary function $f \in Y^X$. For $g \in B$ to be true, we need that $g(x_k) \in U_k$ for each $k \in \{1, \ldots, n\}$. Of course, $f \in B$ as well, so this is the notion for which we say the functions are "close" to eachother.

Note this topology on Y^X is the exact same as the product topology. If we replace X with an indexing set Λ and denote a general element of Λ by λ, then the set $S(\lambda, U)$ simply comprises the functions $\mathbf{x} : \Lambda \to Y$ such that $\mathbf{x}(\lambda) = x_\lambda \in U$. This is precisely $\pi_\lambda^{-1}(U)$ of Y^Λ, and so subbasis elements agree with the product topology.

[7.3.2] Theorem

Let Y^X be a topological space with the topology of pointwise convergence (or otherwise the usual product topology). Then a sequence of functions f_n converges to a function f if and only if, for every $x \in X$, we have that $f_n(x) \to f(x)$. That is, $f_n \to f$ if and only if f_n converges pointwise to f.

[7.3.3] Definition (*Topology of Compact Convergence*)

Let X be a topological space and (Y,d) a metric space. Given some $f \in Y^X$, a compact subset $C \subseteq X$, and some $\varepsilon > 0$, we define

$$B_C(f,\varepsilon) = \left\{ g : \sup_{x \in C}\{d(f(x),g(x))\} < \varepsilon \right\}.$$

The collection of all $B_C(f,\varepsilon)$ forms a basis for a topology on Y^X called the **topology of compact convergence**.

The proof that this forms a basis for a topology is immediate. The idea behind this topology is that a basis element about f admits functions g that are "close" to f on an entire compact set, instead of finitely many points.

[7.3.4] Theorem

Let Y^X be a topological space with the topology of compact convergence. Then a sequence of functions f_n converges to f if and only if $f_n|_C \to f|_C$ uniformly on every compact $C \subseteq X$.

Proof. Recall that we say $f_n \to f$ uniformly on some $A \subseteq X$ if, for any $\varepsilon > 0$, there exists some $N \in \mathbb{Z}_+$ such that $d(f_n(x), f(x)) < \varepsilon$ for every $x \in A$. In other words, the choice of $N \in \mathbb{Z}_+$ must work for all points in A simultaneously, as opposed to at a single point like with uniform convergence. The notion of convergence induced by this topology is that, after fixing some compact set C, $f_n|_C \to f|_C$ if for any basis element $U = B_C(f,\varepsilon)$ (this step involves simply choosing any $\varepsilon > 0$), there is some $N \in \mathbb{Z}_+$ such that $f_n \in U$ for every $n \geq N$. If $f_n \in U$, then we have that $\sup_{x \in C} d(f(x), f_n(x)) < \varepsilon$. Thus $f_n|_C \to f|_C$ is precisely what it means for $f_n|_C \to f|_C$ uniformly.

[7.3.5] Theorem (*Topologies on Function Space*)

Let X be a space and (Y,d) a metric space. For the function space Y^X, one has the following inclusions of topologies:

$$(\text{uniform}) \supseteq (\text{compact convergence}) \supseteq (\text{pointwise convergence}).$$

If X is compact, the first two topologies coincide. If X is discrete, the last two topologies coincide.