8 Baire Spaces

Let's first consider what properties a set with empty interior has (that is, a set whose interior is the empty set). Many consequences come fairly quickly from this. For example, since the interior of a set A is the union of all open subsets of A, we conclude that a set with empty interior contains no open set besides the empty set. As another consequence, every open neighborhood about $x \in A$ must intersect $X \setminus A$, and so $X \setminus A$ is dense in X. There are two perspectives to be taken when considering a set with empty interior: the set itself containing no open set, and the rest of the space being dense.

[8.0.1] Definition (*Empty Interior*)

A set A is said to have empty interior based on the following three equivalent characterizations.

(1) $\mathrm{Int}(A) = \emptyset$.

(2) The only open set A contains is the empty set.

(3) $X \setminus A$ is dense.

[8.0.2] Definition (*Baire Space*)

A space X is said to be a **Baire space** if, for any countable collection $\{A_n\}$ of closed sets with empty interior, we have that $\bigcup A_n$ also has empty interior.

[8.0.3] Theorem (*Characterization of Baire Space*)

A space X is a Baire space if and only if, for any countable collection $\{U_n\}$ of open dense sets, we have that $\bigcap A_n$ is also dense.

[8.0.4] Theorem (*Baire Category Theorem*)

If X is a complete metric or compact Hausdorff space, then it is a Baire space.

Proof. Let $\{A_n\}$ be a countable collection of closed sets with empty interior. We show $\bigcup A_n$ also has closed interior. That is, for any nonempty set U_0 in X, we show there is some $x \in U_0$ such that $x \notin \bigcup A_n$.

Since A_0 has empty interior, note that A_0 cannot contain U_0, and so there is some $x \in U_0$ such that $x \notin A_0$. Note X is regular and A_0 is closed, so there exists an open neighborhood U_1 of x such that

$$\overline{U_1} \cap A_0 = \emptyset,$$
$$\overline{U_1} \subseteq U_0,$$
$$\mathrm{diam}(U_1) < 1 \quad \text{(if } X \text{ is metric).}$$

We can continue this process by induction: any A_k cannot contain U_k, and so there is some $x \in U_k$ such that $x \notin A_k$. Then we can choose an open neighborhood U_{k+1} such that

$$\overline{U_{k+1}} \cap A_k = \emptyset,$$
$$\overline{U_{k+1}} \subseteq U_k,$$
$$\mathrm{diam}(U_{k+1}) < 1/(k+1) \quad \text{(if } X \text{ is metric).}$$

We show $\bigcap \overline{U_n}$ is nonempty. If there is some $x \in \bigcap \overline{U_n}$, then we have that $x \in \overline{U_1} \subseteq U_0$ and $x \notin A_n$ for every n, completing the proof. In the case of X being compact Hausdorff, we note that the decreasing (ordered by inclusion) sequence of closed sets $\{\overline{U_n}\}$ has the finite intersection property, and so by compactness, $\bigcap \overline{U_n}$ is nonempty.

This result holds for complete metric spaces too. Indeed, since $\mathrm{diam}(U_n) \to 0$, we may form a sequence (x_n) such that $x_k \in C_k$ and note that it is Cauchy. By completeness, it converges to some x such that $x \in \overline{C_k}$ for every k, and so $\bigcap \overline{C_k}$ is nonempty. ❦

[8.0.5] Theorem

Any open subset of a Baire space is a Baire space.

[8.0.6] Theorem

Let X be a space and (Y, d) a metric space. Suppose $f_n : X \to Y$ is a sequence of functions such that $f_n \to f$ pointwise. If X is Baire, then the set of points that f is continuous on is a dense set in X.