1 Set Theory and Logic

1.3 Relations

[1.3.1] Definition (*Relation*)

A relation on a set A is some subset $C \subseteq A \times A$. We say that $x, y \in A$ are **related** under C, denoted xCy, if $(x, y) \in C$.

[1.3.2] Example (*Function as Relation*)

Note a function $f : A \to A$ determines a relation

$$\Gamma_f = \{(a, f(a)) : a \in A\} \subseteq A \times A.$$

This relation is unique in that each $a \in A$ is related to precisely one element, namely $f(a)$.

[1.3.3] Definition (*Equivalence Relation*)

A relation $\sim$ on a set A is said to be an **equivalence relation** if the following hold true.

- **Reflexivity.** $x \sim x$ for every $x \in A$.
- **Symmetry.** $x \sim y \iff y \sim x$ for every $x, y \in A$.
- **Transitivity.** $x \sim y$ and $y \sim z \implies x \sim z$ for every $x, y, z \in A$.

[1.3.4] Definition (*Equivalence Class*)

For a set A equipped with an equivalence relation $\sim$, we may write the **equivalence class** of a:

$$[a]_\sim = \{x \in A : a \sim x\}.$$

[1.3.5] Definition (*Partition*)

A partition of some set A is a set $\mathcal{E}$ such that

$$\bigcup_{E \in \mathcal{E}} E = A \qquad \bigcap_{E \in \mathcal{E}} E = \emptyset.$$

That is, $\mathcal{E}$ is a set of disjoint subsets of A that, when combined, form A.

[1.3.6] Theorem (*Equivalence Classes Form a Partition*)

Supose A is a set equipped with an equivalence relation $\sim$. Suppose $E = [x]_\sim$ and $E' = [x']_\sim$ for some $x, x' \in A$. E and E' are either disjoint or equal.

> **Proof.** Suppose $E \cap E' \neq 0$, so there is some $a \in (E \cap E')$. Then $x \sim a$ and $a \sim x'$, so $x \sim x'$. Thus $E \subseteq E'$ (and by symmetry, since $x \sim x'$ means $x' \sim x$), $E' \subseteq E$. Thus $E = E'$. ❦

[1.3.7] Remark (*Equivalence Relations and Partitions*)

As aforementioned, all equivalence relations partition a set. Moreover, a partition is induced by an equivalence relation. Indeed, if $\mathcal{E}$ is a partition of A, we may define $\sim$ such that $x \sim y$ if and only if x and y belong to the same set inside $\mathcal{E}$.

[1.3.8] Definition (*Order Relation*)

A relation C on a set A is called an **order relation** (or a simple/linear order) if it has the following properties.

(1) **Comparability.** If $x, y \in A$ are such that $x \neq y$, then xCy or yCx.

(2) **Nonreflexivity.** For any $x \in A$, xCx is false.

(3) **Transitivity.** For any $x, y, z \in A$, if xCy and yCz, then xCz.

An order relation on a set is typically denoted by $<$.

[1.3.9] Example (*Dictionary Order Relation*)

Suppose A, B are sets equipped with order relations $<_A, <_B$. Define $<$ on $A \times B$ such that, for any $(a_1, b_1), (a_2, b_2) \in A \times B$, it is such that

$$(a_1, b_1) < (a_2, b_2)$$

if $a_1 <_A a_2$, or $a_1 = a_2$ and $b_1 <_B b_2$.

[1.3.10] Definition (*Greatest Lower Bound Property, Least Upper Bound Property*)

An ordered set A is said to have the **least upper bound** property if, for any bounded $A_0 \subseteq A$, the set of upper bounds on A_0 has a least element. The **greatest lower bound** is defined conversely.

1.4 The Integers and Real Numbers

We assume the real numbers exist and satisfy trivial axioms. From the reals we produce the integers.

[1.4.1] Definition (*Inductive Subset, Positive Integers*)

A subset $A \subseteq \mathbb{R}$ is said to be **inductive** if $1 \in A$ and if, for every $x \in A$, the number $x+1 \in A$. Let $\mathcal{A}$ be the collection of all inductive subsets of $\mathbb{R}$. Define the **positive integers** to be the intersection of all elements of $\mathcal{A}$.

$$\mathbb{Z}_+ = \bigcap_{A \in \mathcal{A}} A.$$

Note this implies $\mathbb{Z}_+$ is the smallest inductive subset of $\mathbb{R}$. Note that, of course, $\mathbb{Z}_+$ itself is inductive. Note that if A is any inductive set of positive integers, then it is necessarily $\mathbb{Z}_+$.

We define the set of integers $\mathbb{Z}$ to be the set of positive integers $\mathbb{Z}_+$, 0, and the set of negative integers $-\mathbb{Z}_+$. The set of integers is closed under addition, subtraction, and multiplication, but is not under division—indeed, the set of rationals $\mathbb{Q}$ is said to be the set of all possible quotients of integers.

[1.4.2] Definition (*Section*)

A **section** of the positive integers, denoted S_n, is such that

$$S_{n+1} = \{1, \ldots, n\}$$

for each $n \in \mathbb{Z}_+$.

[1.4.3] Theorem (*Well-Ordering Property*)

Every nonempty subset of $\mathbb{Z}_+$ has a smallest element.

> **Proof.** We first show that for each $n \in \mathbb{Z}_+$, every nonempty subset of the finite segment $\{1, \ldots, n\}$ has a smallest element. For the base case $n = 1$, the only nonempty subset is $\{1\}$ itself, which has 1 as its smallest element. Now, assume the statement holds for n. Let C be a nonempty subset of $\{1, \ldots, n+1\}$. If C consists only of the element $n+1$, then $n+1$ is clearly the smallest element. Otherwise, C must contain elements from $\{1, \ldots, n\}$. In this case, the intersection $C \cap \{1, \ldots, n\}$ is a nonempty subset of $\{1, \ldots, n\}$, so by our inductive hypothesis, it has a smallest element k. Since $k \leq n < n+1$, k is also the smallest element of the entire set C. Thus, by induction, every nonempty subset of any finite segment $\{1, \ldots, n\}$ has a smallest element.
>
> Now let D be any nonempty subset of $\mathbb{Z}_+$. To find its smallest element, choose any element $n \in D$. We only need to look at the elements of D that are less than or equal to n. Let $A = D \cap \{1, \ldots, n\}$. This set A is nonempty and is a subset of a finite segment, so it must have a smallest element k. Because k is the smallest in A, and every element in D that is not in A is strictly greater than n, it follows that k is the smallest element of D. ❦

[1.4.4] Theorem (*Strong Induction Principle*)

Let $A \subseteq \mathbb{Z}_+$. Suppose that for each $n \in \mathbb{Z}_+$, the statement $S_n \subseteq A$ implies $n \in A$. Then $A = \mathbb{Z}_+$.

> **Proof.** Suppose A does not equal $\mathbb{Z}_+$. This means there are some positive integers missing from A. By the Well-Ordering Property, there must be a smallest positive integer n that is not in A. Since n is the smallest element missing, every positive integer less than n must already be in A. This is exactly the condition $S_n \subseteq A$. Our hypothesis states that if $S_n \subseteq A$, then n must be in A. This creates a contradiction: n cannot be both missing from A and required to be in A. Therefore, there can be no missing elements, and $A = \mathbb{Z}_+$. ❦

1.5 Cartesian Products

[1.5.1] Definition (*Indexing Function, Set, and Family*)

Suppose $\mathcal{A}$ is a nonempty collection of sets. An **indexing function** for $\mathcal{A}$ is a surjective function $f : \Lambda \to \mathcal{A}$, where Λ is called the **indexing set**. For each $\lambda \in \Lambda$, we write $A_\lambda = f(\lambda)$. The collection $\{A_\lambda\}_{\lambda \in \Lambda}$ is called the **indexed family of sets** associated with f.

We distinguish the collection $\mathcal{A}$ from the indexed family $\{A_\lambda\}_{\lambda \in \Lambda}$: the latter includes labeling information and may repeat the same set for different indices.

We may use indexing functions to define arbitrary unions and intersections. Suppose $f : \Lambda \to \mathcal{A}$ is an indexing function for some collection of sets $\mathcal{A}$. We may define

$$\bigcup_{\lambda \in \Lambda} A_\lambda = \{x : \text{for at least one } \lambda \in \Lambda, x \in A_\lambda\},$$

$$\bigcap_{\lambda \in \Lambda} A_\lambda = \{x : \text{for every least one } \lambda \in \Lambda, x \in A_\lambda\}.$$

These are, effectively, the unions and intersections of every element of $\mathcal{A}$. For the specific case where $\lambda = \mathbb{Z}_+$, we can denote the indexed family by $\{A_1, \ldots, A_n\}$, while denoting the union and intersection of each A_n by

$$A_1 \cup \cdots \cup A_n \qquad A_1 \cap \cdots \cap A_n.$$

[1.5.2] Definition (*m-Tuple*)

Let $m \in \mathbb{Z}_+$. Given a set X, we define an ***m*-tuple** of the elements of X to be a function

$$\mathbf{x} : \{1, \ldots, m\} \to X.$$

We denote the value of $\mathbf{x}$ at i, $\mathbf{x}(i)$, by x_i and call it the ***i*-th coordinate** of $\mathbf{x}$. We typically denote the entire function by

$$(x_1, \ldots, x_m).$$

[1.5.3] Definition (*Cartesian Product, Revisited*)

Let $\{A_1, \ldots, A_m\}$ be a family of sets indexed by $\{1, \ldots, m\}$. Let $X = A_1 \cup \cdots \cup A_m$. We define the **Cartesian product** of this indexed family, denoted by

$$\prod_{i=1}^{m} A_i \quad \text{or} \quad A_1 \times \cdots \times A_m$$

to be the set of all m-tuples $(x_1, \ldots, x_m)$, where $x_k \in A_k$.

In a similar vein, we define ω-tuples to be m-tuples, simply with their domain equal to $\mathbb{Z}_+$. An ω-tuple is simply an infinite sequence, and the Cartesian product of countably many sets is the set of all ω-tuples, analogous to our definition of Cartesian products of finitely many sets.

1.6 Takeaways

We simply discuss the important ideas of logic and set theory here. In mathematics, there are two parallel ways to define how elements compare to one another: **non-strict** orders (think $\leq$) and **strict** orders (think $<$). We begin by axiomatizing non-strict ordering, discuss the axioms for strict ordering, and highlighting their equivalence.

[1.6.1] Definition (*Partially Ordered Set*)

A set X is said to be a **partially ordered set** (or "poset") if it is equipped with a relation $\leq$ that satisfies the following properties:

(1) **Reflexivity.** For every $x \in X$, $x \leq x$.

(2) **Antisymmetry.** For any $x, y \in X$, if $x \leq y$ and $y \leq x$, then $x = y$.

(3) **Transitivity.** For any $x, y, z \in X$, if $x \leq y$ and $y \leq z$, then $x \leq z$.

[1.6.2] Definition (*Totally Ordered Set*)

A poset X is called a **totally ordered set** (also called a "chain" or "linear order") if every pair of elements is comparable. That is, for any $x, y \in X$, either $x \leq y$ or $y \leq x$.

[1.6.3] Definition (*Strict Partial Order*)

A relation $<$ on a set X is a **strict partial order** if it satisfies:

(1) **Nonreflexivity.** For every $x \in X$, $x < x$ is false.

(2) **Transitivity.** For any $x, y, z \in X$, if $x < y$ and $y < z$, then $x < z$.

[1.6.4] Definition (*Strict Total Order*)

A relation $<$ on a set X is a **strict total order** (often referred to in topology simply as an **order relation**) if it is a strict partial order that also satisfies the law of trichotomy:

3. **Comparability.** For any $x, y \in X$ such that $x \neq y$, either $x < y$ or $y < x$.

[1.6.5] Remark (*The Equivalence of Strict and Non-Strict Orders*)

Strict and non-strict orders are simply two different languages describing the exact same underlying mathematical structure. You can always translate one into the other seamlessly:

From Non-Strict to Strict: Given a non-strict order $\leq$, we induce a strict order $<$ by declaring:

$$x < y \iff x \leq y \text{ and } x \neq y.$$

From Strict to Non-Strict: Given a strict order $<$, we induce a non-strict order $\leq$ by declaring:

$$x \leq y \iff x < y \text{ or } x = y.$$

From now on, we assume partially/totally ordered sets are equipped with strict orders that induce nonstrict orders.

[1.6.6] Definition (*Upper Bound*)

Suppose X is a partially ordered set. For any subset $S \subseteq X$, we say $x \in X$ is an **upper bound** for S if $x \geq s$ for every $s \in S$.

[1.6.7] Definition (*Maximal Element*)

Suppose X is a partially ordered set. For any subset $S \subseteq X$, we say $x \in S$ is a **maximal element** for S if $x \geq s$ for every $s \in S$.

[1.6.8] Axiom (*Zorn's Lemma*)

Suppose X is a non-empty partially ordered set for which every totally ordered subset (i.e., a chain) has an upper bound. Then X has a maximal element.

[1.6.9] Example (*All Vector Spaces Have Bases*)

Essentially, if $X \neq \emptyset$ is a partially ordered set for which every totally ordered subset is bounded above, then X must have one element that is not less than any other element. We cleverly apply this to vector spaces to force the existence of a basis.

Fix V to be an arbitrary vector space, even one that may be infinite-dimensional. Then, we consider the set X of all linearly independent subsets of V. We define a partial ordering on X, such that for any $U, W \in X$, we say

$$U \leq W \iff U \subseteq W.$$

Of course, $X \neq \emptyset$ since the empty set $\emptyset$ is trivially linearly independent, so X is a valid partially ordered set.

To apply Zorn's Lemma, let C be an arbitrary totally ordered subset (a chain) in X. We propose that the union of all sets in C, denoted $M = \bigcup_{U \in C} U$, is an upper bound for C. Clearly, $U \subseteq M$ for all $U \in C$. However, to be a valid upper bound in X, we must verify that $M \in X$; that is, M must be linearly independent.

Suppose for the sake of contradiction that M is linearly dependent. Then, there exists a finite set of vectors $\{v_1, v_2, \ldots, v_n\} \subseteq M$ and scalars $c_1, \ldots, c_n$ (not all zero) such that $c_1 v_1 + \cdots + c_n v_n = 0$. Since M is the union of the chain C, each vector v_i must belong to some set $U_i \in C$. Because C is totally ordered, for any finite collection of sets in C, one of them must contain all the others. Let U_{max} be the largest of these sets $U_1, \ldots, U_n$. This implies that all $v_i \in U_{max}$. But $U_{max} \in X$, meaning it is linearly independent, which contradicts our assumption that these vectors form a linear dependence. Therefore, M must be linearly independent, meaning $M \in X$ and C is indeed bounded above.

By Zorn's Lemma, X contains a maximal element, say B. Because $B \in X$, we already know B is linearly independent. All that remains is to show that B spans V. Suppose it does not; then there exists some vector $v \in V \setminus \mathrm{Span}(B)$. But then the set $B \cup \{v\}$ would be linearly independent, meaning $B \cup \{v\} \in X$. Since $B \subsetneq B \cup \{v\}$, this strictly contradicts the maximality of B. Thus, $\mathrm{Span}(B) = V$, and we conclude that B is a basis for V.

[1.6.10] Axiom (*Axiom of Choice*)

Suppose $\mathcal{C}$ is a collection of non-empty sets. Then you may construct a new set by "choosing" exactly one element from each set.

[1.6.11] Remark

Of course, this seems trivial. If you have infinitely many pairs of shoes, you don't need a special axiom to pick one from each pair (just pick only the left ones). But if you have infinitely many pairs of identical socks, there is no definable rule to pick one from each. The Axiom of Choice simply declares you can do it anyways. It assumes the existence of such a set, even without a definite construction.

[1.6.12] Definition (*Well-Ordered Set*)

A totally ordered set $(X, \le)$ is said to be **well-ordered** if, for any nonempty subset $A \subseteq X$, there exists a strictly least (minimum) element.

[1.6.13] Axiom (*Well-Ordering Axiom*)

Every set can be equipped with a total ordering such that it is well-ordered.

2 Topological Spaces and Continuous Functions

2.1 Topological Spaces

[2.1.1] Definition (*Topology, Topological Space*)

A **topology** on a set X is a collection $\mathcal{T}$ of subsets of X with the following properties.

(1) $\varnothing$ and X are elements of $\mathcal{T}$.

(2) The union of the elements of any subcollection of $\mathcal{T}$ is an element of $\mathcal{T}$.

(3) The intersection of the elements of any finite subcollection of $\mathcal{T}$ is an element of $\mathcal{T}$.

A set X endowed with some topology $\mathcal{T}$ is said to be a **topological space**.

[2.1.2] Definition (*Open Set*)

If $(X, \mathcal{T})$ is some topological space, we say $U \subseteq X$ is **open** if $U \in \mathcal{T}$.

[2.1.3] Remark (*Reformulation of Topological Space*)

With this, we may reformulate a topological space to be a set X together with some collection of subsets, called open sets, such that $\varnothing$ and X are open, as well as the arbitrary union of open sets being open, and the finite intersection of open sets being open.

[2.1.4] Example (*Discrete, Indiscrete/Trivial Topology*)

For any set X, the **discrete topology** is said to be the collection of every single subset of X. The **indiscrete/trivial topology** is said to be the collection containing only $\varnothing$ and X.

[2.1.5] Example (*Finite Complement Topology*)

Fix a set X. Define a topology $\mathcal{T}_f$ such that $U \subseteq X$ is in the topology if $X \setminus U$ is either finite or X. Immediately we see $\varnothing$ and X are elements of the topology, so now we must show the latter two axioms hold true.

Let $\{U_\lambda\}_{\lambda \in \Lambda}$ be an indexed family of open sets. Then note

$$X \setminus \bigcup_{\lambda \in \Lambda} U_\lambda = \bigcap_{\lambda \in \Lambda} (X \setminus U_\lambda).$$

Since each U_λ is open, it follows that the left-hand side is an arbitrary intersection of finite sets, which is finite. Thus the arbitrary union of open sets is open.

Let $U_1, \ldots, U_n$ be finitely many open sets. Then note

$$X \setminus \bigcap_{i=1}^{n} U_i = \bigcup_{i=1}^{n} (X \setminus U_i).$$

Since each U_i is open, it follows that the left-hand side is a finite union of finite sets, which is finite. Thus the finite intersection of open sets is open. Thus $\mathcal{T}_f$ is a well-defined topology.

[2.1.6] Definition (*Finer, Coarser, and Comparable Topologies*)

Suppose X is a set with two topologies, $\mathcal{T}$ and $\mathcal{T}'$. If $\mathcal{T} \subseteq T'$, then $\mathcal{T}$ is **coarser** than $\mathcal{T}'$. If $\mathcal{T} \neq \mathcal{T}'$, then $\mathcal{T}$ is **strictly coarser** than $\mathcal{T}'$. Conversely, $\mathcal{T}'$ is **finer** or **strictly finer** than $\mathcal{T}$. If $\mathcal{T} \subseteq \mathcal{T}'$ or $\mathcal{T}' \subseteq T$, then we say the two topologies are **comparable**.

2.2 Basis for a Topology

[2.2.1] Definition (*Basis, Basis Elements*)

If X is a set, then a **basis** for some topology on X is a collection $\mathcal{B}$ of subsets of X (called **basis elements**) that satisfy the following conditions.

(1) **Covering.** For each $x \in X$, there is some $B \in \mathcal{B}$ such that $x \in B$. That is, $X \subseteq \bigcup_{B \in \mathcal{B}} B$.

(2) **Refinement.** If there exist $B_1, B_2 \in \mathcal{B}$ such that $x \in B_1 \cap B_2$, there is some $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

[2.2.2] Definition (*Topology Generated by Basis*)

If there exists some basis $\mathcal{B}$ for X, we say the topology $\mathcal{T}$ **generated by** $\mathcal{B}$ is as follows:

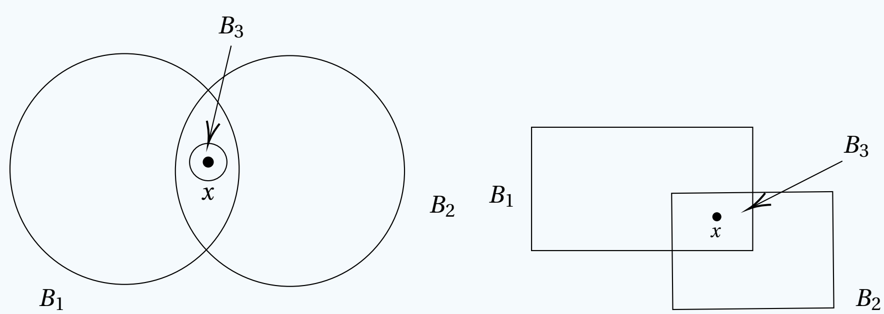
"A subset $U \subseteq X$ is said to be open (i.e., in $\mathcal{T}$) if, for each $x \in U$, there is some $B \in \mathcal{B}$ such that $x \in B \subseteq U$".

[2.2.3] Example (*Basis of Open Balls/Boxes*)

Let $\mathcal{B}$ be the collection of all open balls in $\mathbb{R}^n$. Of course, each $x \in \mathbb{R}^n$ is contained in some open ball $B \in \mathcal{B}$. Now consider some $x \in \mathbb{R}^n$ contained in two open balls $B_1, B_2 \in \mathcal{B}$—that is, $x \in B_1 \cap B_2$. Then we can construct a smaller open ball about x, say B_3, such that $x \in B_3 \subseteq B_1 \cap B_2$ (with $B_3 \in \mathcal{B}$), of course.

We can do an analogous procedure, but instead defining $\mathcal{B}$ to be the collection of all open boxes in $\mathbb{R}^n$. But note that intersections of boxes are, well, boxes, so we do not need to find a basis element inside the intersection of two other basis elements as done with the basis of open balls.

We provide a schematic for these bases in $\mathbb{R}^2$.



[2.2.4] Example (*Basis that Generates Discrete Topology*)

If X is any set, then we can define a basis $\mathcal{B}$ comprising every singleton subset of X. Let $U \subseteq X$ be any arbitrary subset of X. Then note, for each $x \in U$, it's true that $\{x\} \in \mathcal{B}$, such that $x \in \{x\} \subseteq U$. Thus all subsets of X are open, meaning the topology generated by $\mathcal{B}$ is in fact the discrete one.

We now verify that a topology generated by a basis is, indeed, a topology.

[2.2.5] Theorem (*Basis Generates a Topology*)

Suppose $\mathcal{B}$ is a basis for some set X. Let $\mathcal{T}$ be a collection of subsets such that, if some subset $U \subseteq X$ is in $\mathcal{T}$, then for each $x \in U$, there is some $B \in \mathcal{B}$ such that $x \in B \subseteq U$. Then $(X, \mathcal{T})$ forms a topological space.

Proof. First, note $\emptyset \in \mathcal{T}$ vacuously. Next, note that for each $x \in X$, there is some $B_x \in \mathcal{B}$ such that $x \in B_x \subseteq X$ (both by definition of a basis element), so $X \in \mathcal{T}$.

We show arbitrary union is closed in $\mathcal{T}$. Suppose $\{U_\lambda\}_{\lambda \in \Lambda}$ is an indexed family of elements of $\mathcal{T}$. Define

$$U = \bigcup_{\lambda \in \Lambda} U_\lambda.$$

If $x \in U$, then there is some $\lambda \in \Lambda$ such that $x \in U_\lambda$. Since U_λ is open, there exists some $B_\lambda \in \mathcal{B}$ such that $x \in B_\lambda \subseteq U_\lambda \subseteq U$. Thus U is open.

Finally, we show finite intersection is closed in $\mathcal{T}$. That is, for some $n \in \mathbb{Z}_+$, the intersection of open sets $U_1 \cap \cdots U_n$ is open. We prove this inductively, using $n = 2$ as our base case. Suppose $U_1, U_2 \in \mathcal{T}$—we show $U_1 \cap U_2 \in \mathcal{T}$ as well. First, suppose $x \in U_1 \cap U_2$. Then there exist $B_1, B_2 \in \mathcal{T}$ such that $x \in B_1 \subseteq U_1$ and $x \in B_2 \subseteq U_2$. Since $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2 \subseteq U_1 \cap U_2$. Thus $U_1 \cap U_2 \in \mathcal{T}$.

Now suppose this result is true for $n - 1$. Then note

$$U_1 \cap \cdots \cap U_n = \underbrace{(U_1 \cap \cdots U_{n-1})}_{:=U_1} \cap \underbrace{U_n}_{:=U_2}.$$

U_1 is open by hypothesis, of course. Since we have reduced this to $n = 2$, it follows that the intersection is, indeed, finite. Thus $\mathcal{T}$ is a topology! ❦

[2.2.6] Corollary (*Intuition Behind Basis*)

Let X be a set, and let $\mathcal{B}$ be a basis for some topology $\mathcal{T}$ on X. Then $\mathcal{T}$ is the collection of all unions of elements of $\mathcal{B}$.

Proof. Let $\{B_\lambda\}_{\lambda \in \Lambda}$ be a collection of elements of $\mathcal{B}$. Of course, each basis element is an element of the topology it generates. Since topologies are closed under arbitrary union, the union of elements in this collection is a subset of $\mathcal{T}$.

Conversely, suppose $T \in \mathcal{T}$. For each $x \in T$, there is some $B_x \in \mathcal{B}$ such that $x \in B_x \subseteq T$. Thus $T \subseteq \bigcup_{x \in T} B_x$, a union of a collection of elements from $\mathcal{B}$, completing the proof. ❦

Immediately, after choosing a basis for a topology, we note that every element of the topology may be generated by a union of some elements of the basis. This is reminiscent of vectors being generated by a linear combination of some elements of the basis. But note that each vector is uniquely expressible in a vector space's basis, while each element of a topology does not have a necessarily unique expression in the topology's basis.

[2.2.7] Theorem

Suppose $(X, \mathcal{T})$ is a topological space. Let $\mathcal{C}$ be a collection of open sets such that, for each open set $U \subseteq X$ (under the topology $\mathcal{T}$) and for each $x \in U$, there is some $C \in \mathcal{C}$ such that $x \in C \subseteq U$. Then $\mathcal{C}$ is a basis for the topology $\mathcal{T}$.

Proof. First, we show $\mathcal{C}$ is a basis. Note X is open, so by hypothesis, every $x \in X$ is met with some $C \in \mathcal{C}$ such that $x \in C$, and of course $C \subseteq X$. Moreover, suppose $x \in X$ is such that, for some $C_1, C_2 \in \mathcal{C}$, it follows that $x \in C_1 \cap C_2$. Since C_1, C_2 are open under $\mathcal{T}$, it follows that $C_1 \cap C_2$ are open under $\mathcal{T}$, so there exists some $C_3 \in \mathcal{C}$ such that $x \in C_3 \subseteq C_1 \cap C_2$ by hypothesis. Thus $\mathcal{C}$ is a basis.

Now we show the topology generated by $\mathcal{C}$, say $\mathcal{T}'$, is $\mathcal{T}$ itself. This is slightly tautological, but whatever. Suppose $U \subseteq X$ is open under $\mathcal{T}$. The, for each $x \in U$, there is some $C \in \mathcal{C}$ such that $x \in C \subseteq U$ (by hypothesis), meaning $U \in \mathcal{T}'$. Now suppose $W \in \mathcal{T}'$. Then W is expressible as the arbitrary union of elements of $\mathcal{C}$, which are all elements of $\mathcal{T}$, and so $W \in \mathcal{T}$. Thus $\mathcal{T} = \mathcal{T}'$.

It is difficult to describe topologies by their elements (as a topology is a subset of a power set, which is unwieldy). Thus we look towards a topological space's basis, a typically smaller subset of the topology that generates it under unions. We investigate properties of topologies by looking at bases of a topology.

[2.2.8] Theorem

Suppose X is some set with topologies $\mathcal{T}, \mathcal{T}'$, each with a basis $\mathcal{B}, \mathcal{B}'$ respectively. Then the following statements are equivalent.

(1) $\mathcal{T}$ is coarser than $\mathcal{T}'$.

(2) For each $x \in X$ and for every $B \in \mathcal{B}$ such that $x \in B$, there exists some $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$.

Proof.

(2) $\implies$ (1): We seek to show $\mathcal{T} \subseteq \mathcal{T}'$. Suppose $U \in \mathcal{T}$. Then, since $\mathcal{B}$ generates $\mathcal{T}$, there is some $B \in \mathcal{B}$ such that $x \in \mathcal{B} \subseteq U$. By hypothesis, however, there is some $B' \in \mathcal{T}'$ such that $x \in \mathcal{B}' \subseteq U$. Thus $\mathcal{B}'$ generates $\mathcal{T}$, meaning $U \subseteq \mathcal{T}'$.

(1) $\implies$ (2): Suppose $x \in X$ is such that there is some $B \in \mathcal{B}$ for which $x \in B$. Note $\mathcal{B} \subseteq \mathcal{T} \subseteq \mathcal{T}'$, and so $B \in \mathcal{T}'$. Since $\mathcal{T}'$ is generated by $\mathcal{B}'$, there is some $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$.

[2.2.9] Example (*Topologies on* $\mathbb{R}$)

Consider the following topoologies that can be endowed on $\mathbb{R}$.

(1) Let $\mathcal{B}$ be the set of all open intervals in $\mathbb{R}$.

$$\mathcal{B} = \{(a,b) : a < b\} = \{x \in \mathbb{R} : a < x < b\}.$$

The topology generated by $\mathcal{B}$ is said to be the **standard topology** on $\mathbb{R}$. All open sets of $\mathbb{R}$ with respect to this topology take the form of the union of arbitrarily many open intervals.

(2) Now let $\mathcal{B}$ be defined as such.

$$\mathcal{B} = \{[a,b) : a < b\} = \{x \in \mathbb{R} : a \leq x < b\}.$$

The topology generated by $\mathcal{B}$ is said to be the **lower limit topology**. We denote $\mathbb{R}$ endowed with this topology as $\mathbb{R}_\ell$.

(3) Finally, define $K = \{1/n\}_{n \in \mathbb{Z}_+}$, and let $\mathcal{B}$ be defined as such.

$$\mathcal{B} = \{(a,b), (a,b) \setminus K\}.$$

The topology generated by $\mathcal{B}$ is said to be the **K-topology**. We denote $\mathbb{R}$ endowed with this topology as $\mathbb{R}_K$.

[2.2.10] Remark

The topologies on $\mathbb{R}_\ell$ and $\mathbb{R}_K$ (denoted $\mathcal{T}_\ell$ and $\mathcal{T}_K$) are both strictly finer than the topology on $\mathbb{R}$ (denoted $\mathcal{T}$) and both incomparable.

Proof. First, we show $\mathcal{T}_\ell$ is strictly finer than $\mathcal{T}$. Let $x \in (a,b)$, with (a,b) an arbitrary basis element of $\mathcal{T}$. Then note $x \in [x,b) \subseteq (a,b)$, where $[x,b)$ is a basis element of $\mathcal{T}_\ell$, and so $\mathcal{T} \subseteq \mathcal{T}_\ell$ ($\mathcal{T}_\ell$ is finer than $\mathcal{T}$). To show that this is a strict subset, we show $\mathcal{T}_\ell \not\subseteq \mathcal{T}$. Consider some $x \in [x,d)$, with $[x,d)$ a basis element of $\mathcal{T}_\ell$. Then note there is no open interval that contains x and is a subset of $[x,d)$. Thus $\mathcal{T}$ is not finer than $\mathcal{T}_\ell$, meaning $\mathcal{T}_\ell$ is strictly finer than $\mathcal{T}$.

Next, we show $\mathcal{T}_K$ is strictly finer than $\mathcal{T}$. Note the basis of $\mathcal{T}_K$ comprises the basis of $\mathcal{T}$, so it is automatically finer than $\mathcal{T}$—we seek to show $\mathcal{T}$ is not finer than $\mathcal{T}_K$. Let $0 \in (-1,1) \setminus K$, where $(-1,1) \setminus K$ is a basis element of $\mathcal{T}_K$. Then note there is no local refinement in $\mathcal{T}$ that contains 0 (that is, there is no open interval that contains 0 and is a subset of this basis element). Thus $\mathcal{T}$ is not finer than $\mathcal{T}_K$, meaning $\mathcal{T}_K$ is indeed strictly finer than $\mathcal{T}$.

Finally, we show $\mathcal{T}_\ell$ and $\mathcal{T}_K$ are incomparable (i.e., neither is finer than one another). Consider $2 \in [2,3)$, where $[2,3)$ is a basis element of $\mathcal{T}_\ell$. Then note no open interval (a,b)—much less any punctured open interval $(a,b) \setminus K$—contains 2 and is a subset of $[2,3)$. Thus $\mathcal{T}_K$ is not finer than $\mathcal{T}_\ell$. Conversely, consider $0 \in (-1,1) \setminus K$, with $(-1,1) \setminus K$ a basis element of $\mathcal{T}_K$. No left closed, right open interval (i.e., basis elements of $\mathcal{T}_\ell$) contains 0 and is a subset of $(-1,1) \setminus K$. Thus $\mathcal{T}_\ell$ is not finer than $\mathcal{T}_K$, completing the proof. ❦

Consider some arbitrary topological space $(X, \mathcal{T})$, and let $\mathcal{B}$ be a basis for $\mathcal{T}$. Recall that $\mathcal{B}$ must be a cover for X, but it also carries the extra requirement that the intersection of two basis elements must comprise another basis element. In other words, a basis is closed under finite intersection. Then, what is the minimal structure that, when considering the set of all finite intersections, regenerates the basis? We define the subbasis and explain the hierarchy accordingly.

[2.2.11] Definition (*Subbasis*)

A **subbasis** $\mathcal{S}$ for a topology on X is a collection of subsets of X whose union equals X. The topology generated by the subbasis $\mathcal{S}$ is said to be the collection $\mathcal{T}$ of all unions of fintie intersections of elements of $\mathcal{S}$.

[2.2.12] Remark (*Interlude on Subbasis*)

Before we show the topology generated by a subbasis is, indeed, a topology, we provide some form of intuition for it.

A subbasis $\mathcal{S}$ is the most general collection of subsets capable of generating a topology. The primary distinction between a basis and a subbasis lies in the intersection axiom: while a basis $\mathcal{B}$ must be "self-refining"—in the sense that the intersection of any two basis elements must be representable as a union of other basis elements—a subbasis $\mathcal{S}$ is subject to no such constraint.

To bridge this gap, the construction of the topology $\mathcal{T}$ proceeds in two distinct stages:

(1) **Generating the Basis:** We first take the collection of all finite intersections of elements in $\mathcal{S}$. This produces a new collection $\mathcal{B}$ which is, by construction, a basis. Because $\mathcal{B}$ is formed under finite intersections, it satisfies the "self-refining" property that $\mathcal{S}$ lacked.

(2) **Generating the Topology:** We then take the collection of all arbitrary unions of elements in $\mathcal{B}$. This final collection $\mathcal{T}$ now satisfies the remaining axioms of a topology.

Conceptually, this identifies the topology $\mathcal{T}$ as the collection of all arbitrary unions of finite intersections of elements in $\mathcal{S}$. This is the coarsest/minimal topology containing $\mathcal{S}$, allowing one to define a topology by simply declaring a specific collection of sets to be open, bypassing the need to verify that the initial collection satisfies any intersection properties itself.

[2.2.13] Theorem (*Subbasis Generates a Topology*)

Let X be an arbitrary set. The set of all arbitrary unions of the set of all finite intersections of a subbasis forms a topology on X.

Proof. Let X be a set we seek to endow a topology on. We show that the set of all finite intersections of a subbasis forms a basis, and the theorem follows immediately (since we know a basis generates a topology by taking arbitrary unions). Suppose our subbasis $\mathcal{S}$ is written as

$$\mathcal{S} = \{S_\lambda\}_{\lambda \in \Lambda}.$$

We seek to show $\mathcal{B}$, the set of all finite intersections of elements of $\mathcal{S}$, forms a basis. Of course, $\mathcal{S} \subseteq \mathcal{B}$, and $\mathcal{S}$ covers X, so automatically $\mathcal{B}$ covers X as well. Now let

$$B_1 = S_{\lambda_1} \cap \cdots \cap S_{\lambda_m} \qquad B_2 = S'_{\lambda_1} \cap \cdots \cap S'_{\lambda_n},$$

where $B_1, B_2 \in \mathcal{B}$. To complete the proof, we show some finite intersection of S-sets is a subset of $B_1 \cap B_2$. Indeed,

$$B_3 = B_1 \cap B_2 = (S_{\lambda_1} \cap \cdots \cap S_{\lambda_m}) \cap (S'_{\lambda_1} \cap \cdots \cap S'_{\lambda_n})$$

is a finite intersection of sets (by basic properties of sets), so $B_3 \in \mathcal{B}$. Thus, each $x \in (B_1 \cap B_2)$ is such that $x \in B_3 \subseteq (B_1 \cap B_2)$, completing the proof. ❦

2.2.1 Exercises

[2.2.14] PROBLEM

Let $(X, \mathcal{T})$ be a topological space, with $A \subseteq X$. Suppose, for each $x \in A$, there is some open set U_x such that $x \in U_x \subseteq A$. Show A is open.

> **Proof.** Define $O = \bigcup_{x \in A} U_x$. Note that it is an arbitrary union of open sets, and so it is open. Immediately, since each $U_x \subseteq A$, it follows that $O \subseteq A$. But also, since $\{U_x\}_{x \in A}$ forms a cover for A, it's true that $A \subseteq O$. Thus $A = O$, and so A is open.

[2.2.15] PROBLEM

Suppose $\{\mathcal{T}_\alpha\}_{\alpha \in A}$ is a family of topologies on X. Show $\bigcap_{\alpha \in A} \mathcal{T}_\alpha$ is a topology.

> **Proof.** Let $T = \bigcap_{\alpha \in A} \mathcal{T}_\alpha$. Then T is the set of all sets that were open in every topology in $\{T_\alpha\}_{\alpha \in A}$. We note that $\emptyset$ and X are open in each T_α $(\alpha \in A)$, and so they belong in T. Now consider arbitrarily many elements $\{O_\lambda\}_{\lambda \in \Lambda}$ from T. We note these elements are in every $\mathcal{T}_\alpha : \alpha \in A$, so their union should also be in every $\mathcal{T}_\alpha : \alpha \in A$. Thus the union is in T as well. An analogous statement may be made about the intersection of finitely many elements, which completes the proof.

[2.2.16] PROBLEM

Suppose $\mathcal{A}$ is the basis for some topology on X. Show that the topology $\mathcal{A}$ generates is the intersection of all topologies on X that contain $\mathcal{A}$.

> **Proof.** Suppose $\mathcal{A}$ generates a topology T. Let $\{\mathcal{T}_\alpha\}_{\alpha \in A}$ be the family of topologies on X such that $\mathcal{A} \subseteq \mathcal{T}_\alpha$ for each $\alpha \in A$. Let $T' = \bigcap_{\alpha \in A} \mathcal{T}_\alpha$—we show $T = T'$. Of course, $T \in \{\mathcal{T}_\alpha\}_{\alpha \in A}$, and so $T' \subseteq T$. Now we seek to show $T \subseteq T'$. Equivalently, we show that for any open set in T, it follows that aforementioned open set is also in $\mathcal{T}_\alpha$ for every $\alpha \in A$. Fix $\alpha \in A$ and $O \in T$. Note $\mathcal{A}$ is a basis for T, and so for each $x \in O$, there is some $B_x \in \mathcal{A}$ such that $x \in B_x \subseteq O$. Thus $O = \bigcup_{x \in O} B_x$. But note $B_x \in \mathcal{A} \subseteq \mathcal{T}_\alpha$ for every $x \in O$, and so the arbitrary union of each B_x is in $\mathcal{T}_\alpha$. Thus $O \in \mathcal{T}_\alpha$, which completes the proof.

[2.2.17] PROBLEM

Let $\mathcal{B} = \{(a, b) : a < b \in \mathbb{Q}\}$. Show $\mathcal{B}$ is a basis for the standard topology on $\mathbb{R}$.

> **Proof.** Let $O \subseteq \mathbb{R}$ be open with respect to the standard topology. Fix $x \in O$, and choose $\varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \subseteq O$. By the density of $\mathbb{Q}$ in $\mathbb{R}$, there is some $a < b \in \mathbb{Q}$ such that
>
> $$x - \varepsilon < a < x < b < x + \varepsilon.$$
>
> Then note $(a, b) = B \in \mathcal{B}$, and thus $x \in B \subseteq (x - \varepsilon, x + \varepsilon) \subseteq O$. Thus $\mathcal{B}$ is a basis for the standard topology on $\mathbb{R}$.

2.3 Order Topology

If a set is totally ordered, then the ordering actually induces a canonical topology on X called the **order topology**. We investigate its construction and behavior in this section.

First, considering open/closed/half-open intervals in $\mathbb{R}$, we may define them equivalently in an arbitrary totally ordered set, using its order in place of $\mathbb{R}$'s standard ordering. We name them equivalently, suggesting that open intervals in a totally ordered set should be open with respect to the order topology. This is true.

[2.3.1] Definition (*Order Topology*)

Suppose $(X,<)$ is a totally ordered set. Let $\mathcal{B}$ be the collection of all sets of the following forms.

(1) All open intervals $(a,b) \subseteq X$, for every $a,b \in X$.

(2) The half-open interval $[a_0,b) \subseteq X$, for some minimum element (if any) $a_0 \in X$ and any $b \in X$.

(3) The half-open interval $(a,b_0] \subseteq X$, for any $a \in X$ and some maximum element (if any) $b_0 \in X$.

Then $\mathcal{B}$ forms a basis for a topology on X called the **ordered topology**.

> **Proof.** We show $\mathcal{B}$ is, indeed, a basis. First, note that every element of X lies in one of the three prescribed sets: the smallest lies in the upper-open set, the largest lies in the lower-open set, and all other elements lie in an open interval—thus $\mathcal{B}$ covers X as promised. With a long argument by cases, it follows that the intersection of two basis elements can be locally refined. ❦

Note the standard topology on $\mathbb{R}$ is simply the topology induced by the standard order. Note the order topology on $\mathbb{Z}_+$ is simply the discrete topology, since every subset of $\mathbb{Z}_+$ falls into an interval (the singleton $\{n\} = (n-1, n+1)$).

[2.3.2] Definition (*Rays*)

If X is a totally ordered set, an element $a \in X$ determines four rays. They are $(a,\infty), [a,\infty), (-\infty,a), (-\infty,a]$.

[2.3.3] Remark (*Open Rays are Open*)

The rays of form (a,∞) and $(-\infty,a)$ are called **open rays**. Its name is accurate—this interval forms an open set with respect to the order topology. For a set X whose maximum b_0, $(a,\infty) = (a,b_0]$; if no maximum exists, then (a,∞) is simply the union of all basis elements (a,x) for $x : x > a$. Similar argumentation follows for $(-\infty,a)$.

Please note that, for a totally ordered set X with maximum (resp. minimum) b_0 (resp. a_0), then $(a,\infty) = (a,b_0]$ (resp. $(-\infty,a) = [a_0,b)$).

[2.3.4] Theorem (*Open Rays Form Subbasis for Order Topology*)

Suppose X is a totally ordered set. Then the set

$$S = \{(-\infty,a) : a \in X\} \cup \{(a,\infty) : a \in X\}$$

forms a subbasis for the order topology on X.

> **Proof.** It should follow immediately that every basis element can be generated by a finite intersection of one or two open rays. ❦

2.4 Product Topology

[2.4.1] Definition (*Product Topology*)

Suppose X and Y are sets endowed with certain topologies $\mathcal{T}_X, \mathcal{T}_Y$. Define $\mathcal{B}$ to be the collection of sets $U \times V$, where U and V are open subsets of X and Y (with respect to $\mathcal{T}_X$ and $\mathcal{T}_Y$, respectively). Then $\mathcal{B}$ generates the **product topology** on $X \times Y$.

Proof. We show $\mathcal{B}$ is, indeed, a basis. Fix $(x, y) \in (X, Y)$. Since topologies cover their respective set, there exist open subsets U, V of X, Y—thus $(x, y) \in U \times V$, meaning $\mathcal{B}$ covers $X \times Y$. Now let $U_1, U_2 \subseteq X$ and $V_1, V_2 \subseteq Y$ be open. Then

$$(U_1 \times V_1) \cap (U_2 \times V_2) = \underbrace{(U_1 \cap U_2)}_{\in \mathcal{T}_X} \times \underbrace{(V_1 \times V_2)}_{\in \mathcal{T}_Y}.$$

Since the intersection of two open sets is open, we have that the intersection of two basis elements is, indeed, another basis element. ❦

[2.4.2] Theorem (*Product of Bases Generates Product Topology*)

Suppose $\mathcal{B}_X$ and $\mathcal{B}_X$ are bases for topologies on sets X and Y. Then the collection of sets

$$\mathcal{B} = \{B_X \times B_Y : B_X \in \mathcal{B}_X, B_y \in \mathcal{B}_Y\}$$

forms a basis for the product topology of $X \times Y$.

Proof. First, we show $\mathcal{B}$ covers $X \times Y$. Fix $(x, y) \in X \times Y$. Then note $\mathcal{B}_X$ and $\mathcal{B}_Y$ cover X and Y, so there exists $B_X \in \mathcal{B}_X$ and $B_Y \in \mathcal{B}_Y$ such that $(x, y) \in B_X \times B_Y$—thus $\mathcal{B}$ covers $X \times Y$.

Now let $(x, y) \in U \times V \subseteq X \times Y$, where U and V are open subsets of X and Y, respectively. Of course, $U \times V$ is open with respect to the product topology, so we simply have to locally refine it for the specific point (x, y). Note that there is some $B_X \in \mathcal{B}_X$ (resp. $B_Y \in \mathcal{B}_Y$) such that $x \in B_X \subseteq U$ (resp. $y \in B_Y \subseteq V$), and so $(x, y) \in (B_X \times B_Y) \subseteq (U \times V)$. Thus elements of an open subset of $X \times Y$ can be locally refined by $\mathcal{B}$. Thus it is a basis, completing the proof. ❦

[2.4.3] Remark (*Basis for* $\mathbb{R}^2$)

Recall the standard topology on $\mathbb{R}$ is the order topology. Then, in endowing $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$ with the product topology, we seek a basis for $\mathbb{R}^2$. By definition, the basis is simply the product of all open subsets of $\mathbb{R}$. But note, with our previous theorem, we may in fact reduce the basis to simply all sets of form $(a, b) \times (c, d)$, noting that the set of all open intervals of $\mathbb{R}$ forms a basis for the standard topology on $\mathbb{R}$.

[2.4.4] Definition (*Projection Maps*)

Suppose X and Y are sets. Define the following projection maps

$$\pi_X : X \times Y \to X \quad \pi_X(x, y) = x,$$
$$\pi_Y : X \times Y \to Y \quad \pi_Y(x, y) = y.$$

[2.4.5] Theorem (*Subbasis for Product Topology*)

The collection

$$\mathcal{S} = \{\pi_X^{-1}(U) : U \text{ open in } X\} \cup \{\pi_Y^{-1}(V) : V \text{ open in } Y\}$$

forms a subbasis for the product topology on $X \times Y$.

Proof. Note that $\pi_X^{-1}(U) = U \times X$ and $\pi_Y^{-1}(V) = V \times Y$. For any two open subsets $U \subseteq X$ and $V \subseteq Y$, we have that

$$\pi_X^{-1}(U) \cap \pi_Y^{-1}(V) = (U \times Y) \cap (V \times X) = (U \times V) \cap (X \times Y) = U \times V.$$

Thus every element of the standard basis for the product topology on $X \times Y$ is generated by the subbasis. Moreover, since U, V are arbitrary, these are the only types of elements produced by the finite intersection of subbasis elements. Thus the intersection of all subbasis elements produces the standard basis for the product topology. ❦

2.5 Subspace Topology

[2.5.1] Definition

Let $(X, \mathcal{T}_X)$ be a topological space, and let $Y \subseteq X$. Then the collection

$$\mathcal{T}_Y = \{Y \cap U : U \in \mathcal{T}_X\}$$

is the **subspace topology** on Y.

Proof. Note that $\emptyset, X \in \mathcal{T}_X$, and these sets in $\mathcal{T}_X$ correspond directly to $\emptyset, Y$ in $\mathcal{T}_Y$. Now suppose $\{Y_\lambda\}_{\lambda \in \Lambda}$ is a collection of sets in $\mathcal{T}_Y$, where each $Y_\lambda = Y \cap U_\lambda$. Then

$$\bigcup_{\lambda \in \Lambda} Y_\lambda = \bigcup_{\lambda \in \Lambda} Y \cap U_\lambda = Y \cap \underbrace{\left(\bigcup_{\lambda \in \Lambda} U_\lambda\right)}_{\in \mathcal{T}_X},$$

and so $\mathcal{T}_Y$ is closed under arbitrary union. An entirely symmetric argument follows to show $\mathcal{T}_Y$ is closed under finite intersection—thus $\mathcal{T}_Y$ is a topology. ❦

[2.5.2] Theorem

Suppose $(X, \mathcal{T}_X)$ is a topological space with basis $\mathcal{B}$. Then, for any subspace $(Y, \mathcal{T}_Y)$, the collection

$$\mathcal{C} = \{B \cap Y : B \in \mathcal{B}\}$$

forms a basis for $\mathcal{T}_Y$.

Proof. Let $y \in Y$. Then note some $O \in \mathcal{T}_X$ is such that $y \in O \subseteq X$. But then $O \cap Y$ is a basis element, so $y \in (O \cap Y) \subseteq O \subseteq X$. Thus $\mathcal{C}$ covers Y and locally refines every element, making it a basis. ❦

Note that if X is a topological space and $Y \subseteq X$ is endowed with the subspace topology, then any open subset in Y is simply the intersection of some open set in X with Y.

[2.5.3] Corollary (*Subspace Topology is a Subset of Parent Topology for Open Subspaces*)

Suppose $(Y, \mathcal{T}_Y)$ is an **open** subspace of $(X, \mathcal{T}_X)$. Then $\mathcal{T}_Y \subseteq \mathcal{T}_X$. Equivalently, all open sets in Y are open in X.

Proof. Suppose $O \in \mathcal{T}_Y$. Then O is an arbitrary union of basis elements of $\mathcal{T}_Y$. Let $\mathcal{B}$ be a basis for $\mathcal{T}_X$. Then we may write O as the arbitrary union of basis elements, specifically with the basis induced by $\mathcal{B}$. Then

$$O = \bigcup_{\lambda \in \Lambda} Y \cap U_\lambda = \underbrace{Y}_{\in \mathcal{T}_X} \cap \underbrace{\left(\bigcup_{\lambda \in \Lambda} U_\lambda\right)}_{\in \mathcal{T}_X} \in \mathcal{T}_X,$$

where $\{U_\lambda\}_{\lambda \in \Lambda}$ is a set of open sets in X. Then the arbitrary union of each U_λ is open in X, and since Y is open as well, the intersection of this arbitrary union with Y is open as well. Thus O is open in X, meaning $\mathcal{T}_Y \subseteq \mathcal{T}_X$. ❦

[2.5.4] Theorem (*Product Topology and Subspace Topology Coincide*)

Suppose X, Y are topological spaces with subspaces $U \subseteq X$ and $V \subseteq Y$, endowed with the subspace topology. Then the subspace topology on $U \times V \subseteq X \times Y$ is exactly the product topology.

Proof. Let $\mathcal{T}_X, \mathcal{T}_Y, \mathcal{T}_U, \mathcal{T}_V$ be topologies endowed on the respective sets. Let $\mathcal{T}_S$ be the subspace topology for $U \times V \subseteq X \times Y$, and let $\mathcal{T}_{U\times V}$ be the product topology on $U \times V$. Now define these collections of sets as bases for the topologies:

$$\mathcal{B}_S = \{(O_X \times O_Y) \cap (U \times V) : O_X \in \mathcal{T}_X, O_Y \in \mathcal{T}_Y\},$$

$$\mathcal{B}_{U\times V} = \{(O_U \times O_V) : O_U \in \mathcal{T}_U, O_V \in \mathcal{T}_V\}.$$

Let $O_S \in \mathcal{T}_S$. Then note O_S is the arbitrary union of $\{B_{S_\lambda}\}_{\lambda\in\Lambda}$, and so

$$O_S = \bigcup_{\lambda\in\Lambda} (O_{X_\lambda} \times O_{Y_\lambda}) \cap (U \times V) = \bigcup_{\lambda\in\Lambda} \underbrace{\underbrace{(O_{X_\lambda} \cap U)}_{\in \mathcal{T}_U} \times \underbrace{(O_{Y_\lambda} \cap V)}_{\in \mathcal{T}_V}}_{\in \mathcal{T}_{U\times V}} \in \mathcal{T}_{U\times V}.$$

Thus $\mathcal{T}_S \subseteq \mathcal{T}_{U\times V}$. Conversely, suppose $O_{U\times V} \in \mathcal{T}_{U\times V}$. Then

$$O_{U\times V} = \bigcup_{\lambda\in\Lambda} (O_{U_\lambda} \times O_{V_\lambda}) = \left(\bigcup_{\lambda\in\Lambda} (O_{X_\lambda} \times O_{Y_\lambda})\right) \cap (U \times V) \in \mathcal{T}_S,$$

and so $\mathcal{T}_S \supseteq \mathcal{T}_{U\times V}$, completing the proof. ❦

The product topology is nice in this regard, but note the order topology is not. That is, if $(X, <)$ is an ordered set and $(Y, <)$ is a subspace with the same ordering, the order topology is not necessarily the inherited subspace topology. We provide an example where the subspace and order topology do and don't coincide.

[2.5.5] Example (*Subspace Topology vs. Order Topology*)

For our first example, let $Y_1 = [0,1]$ be a subset of $\mathbb{R}$. Consider its subspace and order topologies. The basis for the subspace topology takes the form

$$\mathcal{B}_S = \{(a,b) \cap Y_1 : a, b \in \mathbb{R}\} = \begin{cases} (a,b) & a, b \in Y_1, \\ [0,b) & b \in Y_1, \\ (a,1] & a \in Y_1, \\ Y \text{ or } \emptyset & \text{otherwise.} \end{cases}$$

Note this coincides exactly with the basis for the order topology (excluding $Y, \emptyset$, but they are redundant when generating the topology). Thus the subspace and order topologies are equivalent.

Now let $Y_2 = [0,1) \cup \{2\}$. The basis for the subspace topology takes the form

$$\mathcal{B}_S = \{(a,b) \cap ([0,1) \cup \{2\}) : a, b \in \mathbb{R}\} = \{[(a,b) \cap [0,1)] \cup [(a,b) \cap \{2\}] : a, b \in \mathbb{R}\}.$$

Namely, note that if we choose $(a,b) = (1.5, 2.5)$, we have that $(a,b) \cap Y = \{2\} \in \mathcal{B}_S$, and so it is a member of the subspace topology. But note that any element of the order topology that contains 2 must either be $[0,1) \cup \{2\}$ or $(a,1) \cup \{2\}$, where $0 < a < 1$. In any case, $\{2\}$ is not in the order topology, meaning the order topology and subspace topology do not agree for Y_2.

[2.5.6] Definition (*Convex Set*)

Suppose $(X, <)$ is an ordered set and $(Y, <) \subseteq (X, <)$. We say Y is **convex** in X if, for any $a < b \in Y$, the open interval $(a,b)_X \subseteq Y$.

For example, $[0,1]$ is a convex subset of $\mathbb{R}$, but $[0,1) \cup \{2\}$ is not (since $(1,2)_{\mathbb{R}}$ contains elements like 1.5).

[2.5.7] Theorem (*Order Topology and Subspace Topology Coincide for Convex Subsets*)

Suppose $(X, <)$ is an topological space endowed with the order topology. Let $(Y, <)$ be a subspace of X, where $Y \subseteq X$ is convex. Then the subspace topology and order topology coincide.

Proof. Without loss of generality, suppose Y has a maximum and minimum. Then a basis element for the subspace topology on Y takes the form

$$x \in (\mathcal{B}_S = \{(a,b) \cap Y : a, b \in X\}) = \begin{cases} (a,b) & a, b \in Y, \\ [\min(Y), b) & b \in Y, \\ (a, \max(Y)] & a \in Y, \\ Y \text{ or } \emptyset & \text{otherwise}, \end{cases}$$

which is exactly the order topology. Of course, note that the second and third forms of basis elements are removed if Y does not have a minimum or maximum, respectively. Nevertheless, this coincides perfectly with the order topology on Y, completing the proof. ❦

2.5.1 Exercises

[2.5.8] Problem

Suppose Y is a subspace of X and $A \subseteq Y$. Show that the subspace topology A inherits from Y is the same as the subspace topology A inherits from X.

Proof. Let $\mathcal{T}_{ST}$ denote the subspace topology endowed on set S as a subspace of T. Let $\mathcal{T}_X$ be the topology on X. We show $\mathcal{T}_{AY} \subseteq \mathcal{T}_{AX} \subseteq \mathcal{T}_{AY}$.

First, suppose $O \in \mathcal{T}_{AY}$. Then note there exist a family of open sets $\{O_{Y_\lambda}\}_{\lambda\in\Lambda} \subseteq \mathcal{T}_{YX}$ such that

$$O = \bigcup_{\lambda\in\Lambda}(O_{Y_\lambda} \cap A) = A \cap \left(\bigcup_{\lambda\in\Lambda}(O_{Y_\lambda})\right) = A \cap \left(\bigcup_{\lambda\in\Lambda}(O_{X_\lambda} \cap Y)\right) = \overset{A}{\cancel{(A \cap Y)}} \cap \left(\bigcup_{\lambda\in\Lambda} O_{X_\lambda}\right) \in \mathcal{T}_{AX}.$$

Conversely, let $O \in \mathcal{T}_{AX}$. Then note there exist a family of open sets $\{O_{Y_\lambda}\}_{\lambda\in\Lambda} \subseteq \mathcal{T}_X$ such that

$$O = \bigcup_{\lambda\in\Lambda}(O_{X_\lambda} \cap A) = A \cap \left(\bigcup_{\lambda\in\Lambda}(O_{X_\lambda})\right) = (A \cap Y) \cap \left(\bigcup_{\lambda\in\Lambda}(O_{X_\lambda})\right) = A \cap \left(\bigcup_{\lambda\in\Lambda}(O_{X_\lambda} \cap Y)\right) \in \mathcal{T}_{AY},$$

completing the proof.

[2.5.9] Corollary

Suppose $f : X \to Y$ is a function of sets, and Λ is an index set. Show that

$$f\left(\bigcup_{\lambda\in\Lambda} A_\lambda\right) = \bigcup_{\lambda\in\Lambda} f(A_\lambda).$$

Proof. Suppose $y \in f\left(\bigcup_{\lambda\in\Lambda} A_\lambda\right)$. Then there is some $\mu \in \Lambda$ such that $x \in A_\mu$ and $y = f(x)$. Then, of course, $y \in f(A_\mu) \subseteq \bigcup_{\lambda\in\Lambda} f(A_\lambda)$. Conversely, suppose $y \in \bigcup_{\lambda\in\Lambda} f(A_\lambda)$. There is some $\mu \in \Lambda$ such that there is some $x \in A_\mu$ such that $y = f(x) \in f(A_\mu) \subseteq f\left(\bigcup_{\lambda\in\Lambda} A_\lambda\right)$, completing the proof.

[2.5.10] Problem

A map $f : X \to Y$ is said to be an **open map** if, for every open set $U \subseteq X$, the image $f(U)$ is open in Y. Show $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ are open maps.

Proof. Let U be an open subset of $X \times Y$. Then note there exists an indexing set Λ such that $U = \bigcup_{\lambda\in\Lambda}(O_{X_\lambda} \times O_{Y_\lambda})$. Then

$$\pi_X(U) = \pi_X\left(\bigcup_{\lambda\in\Lambda}(O_{X_\lambda} \times O_{Y_\lambda})\right) = \bigcup_{\lambda\in\Lambda} \pi_X(O_{X_\lambda} \times O_{Y_\lambda}) = \bigcup_{\lambda\in\Lambda} O_{X_\lambda},$$

which is open in X. An analogous procedure can be applied for π_Y.

[2.5.11] Problem

Show that
$$\mathcal{B} = \{(a,b)\times(c,d) : a<b, c<d \text{ and } a,b,c,d\in\mathbb{Q}\}$$
is a basis for the standard topology on $\mathbb{R}^2$.

> **Proof.** Let $O\subseteq\mathbb{R}^2$ be open, and let $x\in O$, where $x=(x_1,x_2)$. Since O is the arbitrary union of open boxes, there is some $a'<b', c'<d'\in\mathbb{R}$ such that $x\in[(a',b')\times(c',d')]\subseteq O$. This means $a'<x_1<b'$ and $c'<x_2<d'$. By the density of the rationals, there exist $a,b,c,d\in\mathbb{Q}$ such that
> $$a'<a<x_1<b<b',$$
> $$c'<c<x_2<d<d'.$$
> Then $x\in[(a,b)\times(c,d)]\subseteq[(a',b')\times(c',d')]\subseteq O$, meaning $\mathcal{B}$ covers $\mathbb{R}^2$ and locally refines each open set, completing the proof.

2.6 Closed Sets and Limit Points

[2.6.1] Definition (*Closed Set*)

A subset A of a topological subspace X is said to be **closed** if its complement $X \setminus A$ is open.

[2.6.2] Example (*Examples of Closed Sets*)

Consider a closed interval $[a, b] \subseteq \mathbb{R}$ equipped with the standard topology. Then note $[a, b]^C = (-\infty, a) \cup (b, \infty)$ is an open set, and so $[a, b]$ is closed.

In general, for some $([x_1, y_1] \times \cdots \times [x_n, y_n]) \subseteq \mathbb{R}^n$, we observe the complement is open, so closed boxes in $\mathbb{R}^n$ are closed (as per the name).

In the topological space endowed with the discrete topology, every subset is open, and so every subset is closed.

[2.6.3] Theorem

Let X be a topological space. Then:

(1) $\emptyset$ and X are closed.

(2) The finite union of closed sets is closed.

(3) The arbitrary intersection of closed sets is closed.

Proof.

(1) Note $\emptyset^C = X$ and $X^C = \emptyset$ are both open, so they're both closed.

(2) Let $A_1, \ldots, A_n$ be finitely many closed sets. Then

$$\left(\bigcup_{k=1}^{n} A_k\right)^C = X \setminus \left(\bigcup_{k=1}^{n} A_k\right) = \bigcap_{k=1}^{n} (X \setminus A_k)$$

is the finite intersection of open sets, which is open. Thus the finite union is closed.

(3) Let $\{A_\lambda\}_{\lambda \in \Lambda}$ be a collection of closed sets. Then

$$\left(\bigcap_{\lambda \in \Lambda} A_k\right)^C = X \setminus \left(\bigcap_{\lambda \in \Lambda} A_k\right) = \bigcup_{\lambda \in \Lambda} (X \setminus A_k)$$

is the arbitrary union of open sets, which is open. Thus the arbitrary intersection is closed. ❦

Now we define a closed set with respect to a topology. If Y is a subspace of X, then a set $A \subseteq Y$ is closed in Y if A is closed in the subspace topology of Y (equivalently, if $Y \setminus A$ is open in Y).

[2.6.4] Theorem (*Criterion for Closed Set in Subspace*)

Let X be a topological space, and let $Y \subseteq X$ be a subspace. Then a subset $A \subseteq Y$ is closed if and only if A is the intersection of a closed subset of X with Y.

> **Proof.**
>
> $\Longrightarrow$: Suppose A is closed in Y. By definition, its complement $Y \setminus A$ is open in Y. By the definition of the subspace topology, there exists an open set U in X such that $Y \setminus A = U \cap Y$. Then,
> $$A = Y \setminus (Y \setminus A) = Y \setminus (U \cap Y) = Y \setminus U = Y \cap (X \setminus U).$$
> Since U is open in X, its complement $X \setminus U$ is closed in X. Letting $C = X \setminus U$, we have $A = C \cap Y$, where C is a closed set in X.
>
> $\Longleftarrow$: Suppose $A = C \cap Y$, where C is a closed set in X. We wish to show that A is closed in Y. Consider the complement of A in Y:
> $$Y \setminus A = Y \setminus (C \cap Y) = Y \setminus C = Y \cap \underbrace{(X \setminus C)}_{\text{open set in } X}.$$
> Since C is closed in X, its complement $X \setminus C$ is open in X. By the definition of the subspace topology, the intersection of an open set in X with Y is open in Y. Therefore, $Y \setminus A$ is open in Y, which implies that A is closed in Y, completing the backward direction. ❦

Note this is analogous to the fact that a set open in a subspace topology is simply an open set in the parent topology intersected with the subspace.

[2.6.5] Theorem

Let X be a topological space, and let $Y \subseteq X$ be a subspace. If a subset $A \subseteq Y$ is closed in Y, and Y is closed in X, then A is closed in X.

> **Proof.** A is the intersection of some closed subset of X with Y, which is also a closed subset of X—thus A is closed in X. ❦

[2.6.6] Definition (*Interior, Closure of Set*)

Let A be a subset of a topological space X. The **interior** of A is said to be the union of all open subsets of A, denoted $\operatorname{Int}(A)$. The **closure** of A is said the be the intersection of all closed subsets of X that contain A, denoted $\overline{A}$.

[2.6.7] Remark (*Relationship between Interior and Closure*)

Note the interior of a subset A of a topological space X is the union of every single open subset of A, which is strictly the largest open subset of A. Analogously, the closure of a subset A of a topological space X is the intersection of all closed subsets of X that contain A, which is strictly the smallest closed superset of A. Then
$$\operatorname{Int}(A) \subseteq A \subseteq \overline{A}.$$
Of course, if A is open, then $\operatorname{Int}(A) = A$, and analogously if A is closed, then $A = \overline{A}$.

Let A be a subset of the topological space X, but also a subset of a subspace $Y \subseteq X$. Of course, the closure of A as a subset of Y is not necessarily the same as the closure of A as a subset of X.

[2.6.8] THEOREM (*Closure of Subset under Subspace*)

Suppose X is a topological space, Y is a subspace of X, and $A \subseteq Y \subseteq X$. Then the closure of A under Y, say $\mathrm{Cl}_Y(A)$ equals $\mathrm{Cl}_X(A) \cap Y$.

Proof. We show $\mathrm{Cl}_Y(A) = \mathrm{Cl}_X(A) \cap Y$ by proving a forward and backward inclusion.

$\subseteq$: Suppose $x \in \mathrm{Cl}_Y(A)$. Then note x is in every closed superset of A contained in Y, and of course $\mathrm{Cl}_X(A) \cap Y$ is one example, so $x \in \mathrm{Cl}_X(A) \cap Y$.

$\supseteq$: Suppose $x \in \mathrm{Cl}_X(A) \cap Y$. Then let C be any arbitrary closed superset of A such that $A \subseteq C \subseteq Y$. If $x \in C$, then it follows that $x \in \mathrm{Cl}_Y(A)$. Since C is closed in Y, and Y is a subspace of X (and thus endowed with the subspace topology), there is some closed subset $C_X \subseteq X$ such that $C = C_X \cap Y$. But $A \subseteq (C = C_X \cap Y)$, implying $A \subseteq C_X$. But $\mathrm{Cl}_X(A)$ is the minimal superset of A that is closed, so $A \subseteq \mathrm{Cl}_X(A) \subseteq C_X$. Thus $x \in C$, completing the proof. ❦

The definition of the closure of a set, while elegant, is impractical for explicit computation. For a topological space X and subset A, one would need to identify every closed subset of X containing A, then intersect them all—a collection far too large to work with directly. Instead, we may characterize membership in $\overline{A}$ in a more tractable ways: via the concept of *limit points*.

[2.6.9] DEFINITION (*Neighborhood*)

A set U is said to be a **neighborhood** containing x if U is an open set containing x.

[2.6.10] THEOREM (*Characterization of Closure of Set*)

Suppose A is a subset of the topological space X. Then the following statements are true.

(1) $x \in \overline{A}$ if and only if every neighborhood U containing x intersects A.

(2) $x \in \overline{A}$ if and only if each $B \in \mathscr{B}$—where $\mathscr{B}$ is a basis for the topology of X—containing x intersects A.

Proof.

(1) We proceed by contrapositive. That is, we show that $x \notin \overline{A}$ if and only if there exists some open neighborhood $U \subseteq X$ of x. First, suppose $x \notin \overline{A}$. Note $X \setminus \overline{A}$ is an open subset of X that must contain x, but does not intersect A by construction. Conversely, suppose some open neighborhood $U \subseteq X$ of x exists that doesn't intersect A. Then $X \setminus U$ is a closed superset of A that does not contain x, and so $x \notin \overline{A}$, completing the proof.

(2) Basis elements that contain x are open neighborhoods about x, and open neighborhoods about x that intersect A contain basis elements.

❦

[2.6.11] REMARK (*Neighborhood Characterization of Closure*)

From real analysis, recall the open interval $A = (0,1) \subseteq \mathbb{R}$ (with $\mathbb{R}$ endowed with the standard topology), and note $\overline{A} = [0,1]$. Now consider $0 \in \overline{A}$. (1) from the previous theorem says that all neighborhoods about 0 must intersect A. Indeed, any open interval $(-\varepsilon, \varepsilon) \cap A \neq \emptyset$, and so 0 is in the closure for A.

For some set A, our theorem says that the closure $\overline{A}$ contains A, as well as additional points that, when taking an open neighborhood around, always intersect A. We formalize this idea and relate them back to the closure.

[2.6.12] Definition (*Limit Point*)

Suppose A is a subset of a topological space X. We say $x \in X$ is a **limit point** if every open neighborhood $U \subseteq X$ about x intersects A at some point *other than x itself*. That is, for every open set $U \subseteq X$ with $x \in U$, it follows that $U \cap (A \setminus \{x\}) \neq \emptyset$. Said differently, x is a limit point if it belongs to the closure of $A \setminus \{x\}$.

[2.6.13] Theorem (*Closure and Limit Points*)

Let A be a subset of the topological space X. Let A' be the set of all limit points of A. Then

$$\overline{A} = A \cup A'.$$

Proof. We prove equality by forward and backward inclusion.

$\subseteq$: Suppose $x \in \overline{A}$. If $x \in A$, then $x \in A \cup A'$. Otherwise, if $x \notin A$, then $A = A \setminus \{x\}$. But note that, since $x \in \overline{A}$, every open neighborhood $U \subseteq X$ of x intersects A nontrivially, so $U \cap (A \setminus \{x\}) \neq \emptyset$. Thus $x \in A' \subseteq A \cup A'$.

$\supseteq$: Suppose $x \in A \cup A'$. If $x \in A$, $x \in \overline{A}$, obviously. Now suppose $x \notin A'$, and so $x \in A'$. Analogous to before, this means $A \setminus \{x\} = A$. Since $x \in A'$, for every open neighborhood $U \subseteq X$ of x, $U \cap (A \setminus \{x\}) = U \cap A \neq 0$. Thus $x \in \overline{A}$, completing the proof. ❦

[2.6.14] Corollary

A subset of a topological space is closed if and only if it contains its limit points.

Proof. Let A be a subset of topological space X, and let A' be the set of limit points. First, suppose A is closed. Then $A = \overline{A} = A \cup A'$, meaning $A' \subseteq A$. Now suppose A contains all its limit points, and so $A' \subseteq A$. Then $A' \cup A \subseteq A$ and $A \subseteq A' \cup A$, meaning $A = A \cup A' = \overline{A}$, completing the proof. ❦

[2.6.15] Remark (*Prelude to Hausdorffness*)

Recall the standard topology on $\mathbb{R}$. A few nice properties are evident in its study.

(1) For each $x_0 \in \mathbb{R}$, the singleton $\{x_0\}$ is closed. For any $x : x \neq x_0$, we may construct an open neighborhood $U = (x-\varepsilon, x+\varepsilon)$ specifically to enforce $x_0 \notin U$ by choosing $\varepsilon > 0$ to be small enough.

(2) For any convergent real sequence $(x_n) \subseteq \mathbb{R}$, we say $x_n \to x$ if, for every open neighborhood U of x, there is some $N \in \mathbb{Z}_+$ such that $x_n \in U$ for every $n \geq N$. In $\mathbb{R}$, this limit is unique—that is, if $x_n \to x_1$ and $x_n \to x_2$, then $x_1 = x_2$.

These don't transfer to general topological spaces though. Consider the triplet $S = \{a, b, c\}$ endowed with topology $\{\emptyset, \{b\}, \{a, b\}, \{b, c\}, S\}$.

(1) We see that $\{b\}$ is not closed, since $S \setminus \{b\} = \{a, c\}$ is not in the topology (i.e., not open).

(2) Moreover, with $\mathbb{R}$'s definition of convergence, let's define $x_n = b$ and see what it converges to. For every open neighborhood of b ($\{a, b\}, \{b\}, \{b, c\}, \{a, b, c\}$), we see $x_n = b$ is in each neighborhood for every $n \in \mathbb{Z}_+$. The open neighborhoods of a ($\{a, b\}, \{a, b, c\}$) and c ($\{b, c\}, \{a, b, c\}$) also contain b. Thus the conclusion is that $\lim(x_n) \in \{a, b, c\}$, which does not agree with $\mathbb{R}$'s topological properties.

There is some property of $\mathbb{R}$'s standard topology that makes it nicer than S's topology, in that singletons are closed and convergent sequences have a unique limit.

[2.6.16] Definition (*Hausdorffness*)

A topological space X is said to be **Hausdorff** if, for any pair of distinct points $x_1, x_2 \in X$, there exists two disjoint neighborhoods $U_1, U_2 \subseteq X$ about x_1 and x_2 respectively.

[2.6.17] Theorem (*Singletons are Closed in Hausdorff Space*)

Every singleton subset of a topological space X is closed.

> **Proof.** Suppose $\{x_0\} \subseteq X$. Let $x \in X$ be such that $x \neq x_0$. By Hausdorffness of X, there are disjoint open neighborhoods $U_1, U_2 \subseteq X$ about x and x_0. Thus $x \notin \overline{A}$. This implies $\overline{A} = \{x_0\}$, meaning the singleton is closed. ❦

The property that singleton subsets of a topological space are closed is weaker than the space being Hausdorff. We extract this condition as the T_1 **axiom**.

[2.6.18] Definition (*T_1 Space*)

A topological space X is said to be T_1 if every singleton subset of X is closed.

From a geometric perspective, a space is T_1 if for any two distinct points x, y, there exists an open neighborhood of x that does not contain y. The space is Hausdorff if, for any two distinct points x and y, there exist disjoint open neighborhoods of x and y. Thus Hausdorffness is a stronger condition than T_1. We will see shortly that although T_1 spaces preserve our intuition about limit points, T_1 alone is not strong enough to guarantee the uniqueness of sequence limits, whereas Hausdorff spaces do.

[2.6.19] Theorem

Let A be a subset of the topological space X satisfying the T_1 axiom. Then x is a limit point of A if and only if every neighborhood of X contains infinitely many points of A.

Proof.

($\Longrightarrow$): Suppose x is a limit point of A. Then there is some open neighborhood $U \subseteq X$ of x such that $(U \cap (A \setminus \{x\}) = S) \neq \emptyset$. Suppose S is finite, so $S = \{x_1, \ldots, x_n\}$. Note that S is a finite union of singleton subsets of X, and since singletons are closed in T_1 spaces, S is closed. Thus $X \setminus S$ is open. Let $U' = U \cap (X \setminus S)$. Then U' is open, $x \in U$, but $U \cap (A \setminus \{x\}) = \emptyset$. We have constructed an open neighborhood about x that does not intersect $A \setminus \{x\}$, meaning x is not a limit point, forcing a contradiction. Thus $U \cap (A \setminus \{x\})$ is infinite.

($\Longleftarrow$): Every open neighborhood about x intersects A infinitely amount of times, so every open neighborhood about x intersects $A \setminus \{x\}$ infinitely many times, meaning x is a limit point. ❦

[2.6.20] Theorem (*Uniqueness of Limits in Hausdorff Space*)

Suppose X is a Hausdorff topological space. Then a sequence $(x_n)_{n=1}^{\infty} \subseteq X$ converges to, at most, one limit point.

Proof. Suppose x_n converges, so $x_n \to x$. Let $x_0 \neq x$: we show $x_n \not\to x_0$, confirming x is the only limit for the sequence. Since $x \neq x_0 \in X$, a Hausdorff space, we can construct disjoint open neighborhoods U and U_0 about x and x_0 respectively such that $U \cap U_0 = \emptyset$. Recall that since $x_n \to x$, there exists some $N \in \mathbb{Z}_+$ such that $x_n \in U$ for every $n \geq N$. But since U_0 is disjoint from U, it follows that the only terms of the sequence U_0 can possibly contain are $x_1, \ldots, x_{N-1}$. In other words, the tail for (x_n) is not contained in U_0, so $x_n \not\to x_0$, completing the proof. ❦

2.6.1 Problems

[2.6.21] Problem

Let X be an ordered set endowed with the order topology. Show that $\overline{(a,b)} \subseteq [a,b]$. Under what conditions does equality hold?

Proof. Note $\overline{(a,b)}$, by definition, is the smallest closed superset of (a,b). Thus, if $[a,b]$ is a closed superset of (a,b), then $\overline{(a,b)} \subseteq [a,b]$ follows immediately. Note that $[a,b]^c = (\leftarrow, a) \cup (b, \rightarrow)$ is a union of two open sets (so it's open), making $[a,b]$ closed, completing the proof.

Equality is achieved when $[a,b] \subseteq \overline{(a,b)}$, since we have forward and backward inclusion. Since $[a,b] = (a,b) \cup \{a\} \cup \{b\}$, simply knowing that a,b are limit points of (a,b) forces equality. We show a (resp. b) is a limit point of (a,b) if a (resp. b) has no immediate successor (resp. immediate predecessor).

Suppose a has no immediate successor. Let $U \subseteq X$ be an open neighborhood about a. Then note that we can locally refine U by choosing some basis element of the order topology, V, such that $a \in V \subseteq U$. Note a basis element is of the form $V = (a,c)$, where $c : a < c \leq b$. In any case, since a has no immediate successor, there is some $x < c$ for which $x \in V \subseteq U$. This means $x \in U \cap ((a,b) \setminus \{a\}) = U \cap (a,b)$, meaning a is a limit point for (a,b).

[2.6.22] Problem

Let A, B be subsets of a topological space X. Prove the following.

(1) If $A \subseteq B$, then $\overline{A} \subseteq \overline{B}$.

Proof.

(1) Suppose $A \subseteq B$. Let $x \in \overline{A}$. If $x \in A$, then $x \in B \subseteq \overline{B}$ trivially, so suppose $x \notin A$. Then x is a limit point, so every open neighborhood $U \subseteq X$ about x is such that $U \cap (A \setminus \{x\}) \neq \emptyset$. But $A \subseteq B$, so this implies $U \cap (B \setminus \{x\}) \neq \emptyset$, meaning x is a limit point of B too, so $x \in \overline{B}$.

[2.6.23] Problem

Show that every order topology is Hausdorff.

Proof. Suppose an ordered set X is endowed with the order topology. Let $x \neq y \in X$, and without loss of generality, suppose $x < y$. Suppose y is x's immediate successor: then there exist open neighborhoods $U = (-\infty, y) \ni x$ and $V = (x, \infty) \ni y$ where $U \cap V = \emptyset$. If y is not x's immediate successor, there exists $t : x < t < y$. Then $U = (-\infty, t) \ni x$ and $V = (t, \infty) \ni y$ such that $U \cap V = \emptyset$, completing the proof.

[2.6.24] Problem

Show the product of two Hausdorff spaces is Hausdorff.

Proof. Let $(x_1, y_1) \neq (x_2, y_2) \in X \times Y$, where X and Y are Hausdorff topological spaces. Without loss of generality, suppose $x_1 \neq x_2$. By Hausdorffness of X, there exists open neighborhoods $U_X \ni x_1$ and $V_X \ni x_2$ such that $U_X \cap V_X = \emptyset$. There also exists some open neighborhood $V \ni y_1, y_2$. Then $(U_X \times V) \ni (x_1, y_1)$ and $(V_X \times V) \ni (x_2, y_2)$ are disjoint open neighborhoods of $X \times Y$, completing the proof.

[2.6.25] Problem

Show the subspace of a Hausdorff space is Hausdorff.

Proof. Suppose X is a Hausdorff space and $Y \subseteq X$ is a subspace. Let $x \neq y \in Y$. Note there exists open neighborhoods $U \subseteq X$ [illegible] and $V \cap Y$. Of course, $x \in U \cap Y$ and $y \in V \cap Y$, and note $(U \cap Y) \cap (V \cap Y) = (U \cap V) \cap Y = \emptyset \cap Y = \emptyset$, completing the proof. ❦

[2.6.26] Problem

Show that a topological space X is Hausdorff if and only if the diagonal $\Delta = \{(x,x) : x \in X\}$ is closed in $X \times X$.

Proof.

($\Longrightarrow$): Suppose X is Hausdorff. Suppose $(x,y) \in X \times X$ is a limit point for Δ. Then, for every open neighborhood $U \subseteq X \times X$ of (x,y), it follows that $U \cap (\Delta \setminus (x,y)) \neq \emptyset$. For the sake of contradiction, suppose $x \neq y$. Then, by Hausdorffness of X, there exist disjoint open neighborhoods $U \subseteq X$ and $V \subseteq X$ such that $x \in U$ and $y \in V$. Note then that $U \times V$ is open in $X \times X$. Since U and V are disjoint, there are no tuples with the same entry in each slot when taking their Cartesian product—that is, $(U \times V) \cap (\Delta \setminus (x,y)) = \emptyset$, which is a contradiction. Thus $x = y$, implying Δ contains all its limit points and thus is closed.

($\Longleftarrow$): Suppose Δ is closed, and so Δ^c is open. Let $x \neq y \in X$, then observe $(x,y) \in \Delta^c$. Since Δ^c is an open subset of $X \times X$, there is some basis element $U \times V$ (where U, V are open subsets of X) such that $(x,y) \in U \times V \subseteq \Delta^c$. For $U \times V$ to be a subset of Δ^c, this implies no elements of $U \times V$ may be in Δ: thus $U \cap V = \emptyset$. Thus we have constructed two disjoint open neighborhoods that contain x and y respectively, completing the proof. ❦

[2.6.27] Problem

Endow $\mathbb{R}$ with the finite complement topology. For what points of $\mathbb{R}$ does $x_n = 1/n$ converge to.

Proof. Note a set $O \subseteq \mathbb{R}$ is open if $O = \emptyset$, or $\mathbb{R} \setminus O$ is finite (i.e., it misses finitely many points of $\mathbb{R}$). Let $x \in \mathbb{R}$ be arbitrary, and let $O \subseteq \mathbb{R}$ be an arbitrary open neighborhood such that $x \in O$. Note O can only miss finitely many elements of $\mathbb{R}$, so O must contain infinitely many elements of x_n. That is, there is some $N \in \mathbb{Z}_+$ such that $x_n \in O$ for every $n \geq N$. Thus $x_n \to x$, but x is arbitrary, so (x_n) converges to every real number. ❦

[2.6.28] Definition (*Boundary*)

Suppose X is a topological space and $A \subseteq X$. We define the boundary of A to be the set

$$\partial A = \overline{A} \cap \overline{X \setminus A}.$$

[2.6.29] Remark (*Intuition Behind Boundary*)

Suppose X is a topological space and $A \subseteq X$. Recall that if $x \in \overline{A}$, then every open neighborhood $U \subseteq X$ about x is such that $U \cap A \neq \emptyset$. Equivalently, if $x \in \overline{X \setminus A}$, then every open neighborhood $U \subseteq X$ about x is such that $U \cap (X \setminus A) \neq \emptyset$. Thus, if $x \in (\overline{A} \cap \overline{X \setminus A}) = \partial A$, every open neighborhood $U \subseteq X$ about x must have nontrivial intersection with A and $X \setminus A$. Graphically, drawing an open neighborhood about a point leads to it spilling inside and outside the set.

[2.6.30] Problem

Suppose X is a topological space and $A \subseteq X$. Show that $\mathrm{Int}(A)$ and ∂A are disjoint.

> **Proof.** Suppose $x \in \mathrm{Int}(A)$. Since $\mathrm{Int}(A) \subseteq A$, it follows that $x \in A$. For the sake of contradiction, suppose $x \in \partial A$, implying $x \in \overline{X \setminus A}$. Note $X \setminus A = (X \setminus A) \cap (X \setminus A)'$ (where S' refers to the limit points of S). Since $x \in A$, $x \notin (X \setminus A)$, so it is only possible that $x \in (X \setminus A)'$. This implies that every open subset $U \subseteq X$ must be such that $U \cap (A^c \setminus \{x\}) \neq \emptyset$. But note $x \in \mathrm{Int}(A) \subseteq A$, meaning every open neighborhood of x must be fully contained in A, which is a contradiction. Thus $\mathrm{Int}(A) \cap \partial A = \emptyset$. ❦

[2.6.31] Problem

Suppose X is a topological space and $A \subseteq X$. Show $\overline{A} = \mathrm{Int}(A) \cup \partial A$. That is, $\overline{A}$ is the disjoint union of the interior and boundary of A.

> **Proof.** We prove this by forward and backward inclusion. First, suppose $x \in \overline{A}$, and suppose $x \notin \mathrm{Int}(A)$. This implies there are no X-open neighborhoods of x that are subsets of A. Indeed, this implies the only X-open neighborhoods of x must intersect $X \setminus A$. Thus $x \in \overline{X \setminus A}$, so $x \in \partial A$. Thus every $x \in \overline{A}$ must be either in $\mathrm{Int}(A)$ or ∂A, and so $\overline{A} \subseteq \mathrm{Int}(A) \cup \partial A$. Conversely, suppose $x \in \mathrm{Int}(A) \cup \partial A$. If $x \in \mathrm{Int}(A)$, then $x \in A \subseteq \overline{A}$. If $x \in (\partial A = \overline{A} \cap \overline{X \setminus A})$, then $x \in \overline{A}$, completing the proof. ❦

[2.6.32] Problem

Suppose X is a topological space and $A \subseteq X$. Show that $\partial A = \emptyset$ if and only if A is clopen.

> **Proof.** Suppose A is X-clopen. Since A is closed, $A = \overline{A}$. Let $x \in A = \overline{A}$ be a limit point of A. Then note every X-open neighborhood $O \ni x$ is such that $O \subseteq A$, so no open neighborhoods intersect $X \setminus A$. Thus $x \notin \overline{X \setminus A}$, implying $\partial A = \emptyset$. Conversely, suppose $\partial A = \overline{A} \cap \overline{X \setminus A} = \emptyset$. If x is a limit point of A, then $x \in \overline{A}$, then it cannot be in $\overline{X \setminus A}$ and so $x \in A$, meaning A is closed. The same argument can be made to show $X \setminus A$ is also closed, meaning A is open as well, completing the proof. ❦

[2.6.33] Problem

Suppose X is a topological space and $A \subseteq X$. Show that U is open if and only if $\partial U = \overline{U} \setminus U$.

Proof. ($\Longrightarrow$): Suppose U is open. We prove the forward direction by forward and backward inclusion. To get started, suppose $x \in \overline{U} \setminus U$. Then note $x \notin U$, and so $x \in X \setminus U \subseteq \overline{X \setminus U}$, and so $x \in \partial U$. Conversely, suppose $x \in \partial U$. Immediately, we know $x \in \overline{U}$. For the sake of contradiction, suppose $x \in U$. Since U is open, there is some basic X-open neighborhood O such that $x \in O \subseteq U$. But $O \cap (X \setminus U) = \emptyset$, so $x \notin \overline{X \setminus U}$, a contradiction. Thus $x \notin U$, meaning $x \in \overline{U} \setminus U$. Thus $\partial U = \overline{U} \cap \overline{X \setminus U} = \overline{U} \setminus U$.

($\Longleftarrow$): Suppose $\partial U = \overline{U} \setminus U$. To show U is open, we show $X \setminus U$ is closed, or that $X \setminus U = \overline{X \setminus U}$. Let $x \in \overline{X \setminus U}$ be a limit point for $X \setminus U$. Note that if $x \in U$, then $x \in \overline{U}$ and so $x \in \overline{U} \cap \overline{X \setminus U}$, but $x \notin U \setminus \overline{U}$, which is contradictory—thus $x \in X \setminus U$, completing the proof. ❦

2.7 Continuous Functions

[2.7.1] Definition (*Continuous Function*)

A set-function $f : X \to Y$ is said to be **continuous** if, for every open subset $V \subseteq Y$, the preimage $f^{-1}(V)$ is an open subset of X.

Note continuity depends on the topologies endowed on the domain and codomain of the function.

[2.7.2] Remark (*Continuity of Basis and Subbasis Elements*)

Suppose the topology on Y has a basis $\mathcal{B}$. Then any open subset $V \subseteq Y$ can be written as

$$V = \bigcup_{\lambda \in \Lambda} B_\lambda,$$

where $\{B_\lambda\}_{\lambda \in \Lambda}$ is an arbitrary collection of basis elements. Then note

$$f^{-1}(V) = f^{-1}\left(\bigcup_{\lambda \in \Lambda} B_\lambda\right) = \bigcup_{\lambda \in \Lambda} f^{-1}(B_\lambda),$$

so if the preimage of each basis element is open, then the function is continuous on its entire domain. Similarly, note that $\mathcal{B}$ can have a subbasis $\mathcal{S}$, so any B_λ can be written as

$$B_\lambda = \bigcap_{k=1}^{n} S_k,$$

where $\{S_1, \ldots, S_n\} \subseteq \mathcal{S}$ is a collection of subbasis elements. Then

$$f^{-1}(B_\lambda) = f^{-1}\left(\bigcap_{k=1}^{n} S_k\right) = \bigcap_{k=1}^{n} f^{-1}(S_k),$$

so if the preimage of each subbasis element is open, then the function is continuous on its entire domain.

[2.7.3] Example (*Continuity of Single-Variable Real Functions*)

Consider some function $f : \mathbb{R} \to \mathbb{R}$, where the domain and codomain are endowed with the standard topology.

First, suppose f is continuous with our definition utilizing the preimage of open sets. Fix some $x_0 \in \mathbb{R}$ and let $\varepsilon > 0$ be arbitrary. Then note $V = (f(x_0) - \varepsilon, f(x_0) + \varepsilon)$ is a basis element for $\mathbb{R}$, so it is open in the codomain. Of course, $x_0 \in f^{-1}(V)$. Since $f^{-1}(V)$ is an open subset of $\mathbb{R}$, we can locally refine by some basis element (a, b) such that $x_0 \in (a, b) \subseteq f^{-1}(V)$. Thus there is some $\delta > 0$ such that $(x_0 - \delta, x_0 + \delta) \subseteq f^{-1}(V)$. Indeed, this implies the $\varepsilon - \delta$ variant of continuity at x_0: for any $\varepsilon > 0$, there is some $\delta > 0$ such that for every $x \in (x_0 - \delta, x_0 + \delta)$, it is true that $f(x) \in (f(x_0) - \varepsilon, f(x_0) + \varepsilon)$.

Now suppose f is continuous with the $\varepsilon - \delta$ definition. That is, for any $x_0 \in \mathbb{R}$ and $\varepsilon > 0$, there is some $\delta > 0$ such that

$$|x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon.$$

This can be reformulated into open and closed sets quite easily:

$$x \in B_\delta(x_0) \implies f(x) \in B_\varepsilon(f(x_0)).$$

Thus f is continuous if every open set containing $f(x_0)$ has a preimage that is an open set. Thus these two characterizations are equal.

[2.7.4] Theorem (*Characterizations of Continuity*)

Let X, Y be topological spaces, and let $f : X \to Y$. Then the following are equivalent.

(1) f is continuous.

(2) For every subset $A \subseteq X$, it follows that $f(\overline{A}) \subseteq \overline{f(A)}$.

(3) For every closed subset $B \subseteq Y$, the preimage $f^{-1}(B) \subseteq X$ is closed as well.

(4) For every $x \in X$ and for every open neighborhood $V \subseteq Y$ of $f(x)$, there is some open neighborhood $U \subseteq X$ of x where $f(U) \subseteq V$.

Proof. We prove (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (1) and (1) $\Longrightarrow$ (4) $\Longrightarrow$ (1).

(1) $\Longrightarrow$ (2): Suppose f is continuous. We seek to show that, for any $x \in \overline{A}$, it is true that $f(x) \in \overline{f(A)}$. To get started, consider an open neighborhood $V \subseteq Y$ about $f(x) \in f(\overline{A})$. Then note $f^{-1}(V) \subseteq X$ is open by continuity. Since $x \in \overline{A}$, all open neighborhoods of x intersect A. Since $x \in f^{-1}(V)$ and $f^{-1}(V)$ is open, it follows that there is some $y \in f^{-1}(V) \cap A$. Then note $f(y) \in V \cap f(A)$. Thus arbitrary open neighborhoods of $f(x)$ intersect $f(A)$, implying $f(x)$ is in the closure of $f(A)$. Thus $f(x) \in \overline{f(A)}$, and ultimately $f(\overline{A}) \subseteq \overline{f(A)}$.

(2) $\Longrightarrow$ (3): Suppose that, for every $A \subseteq X$, it is true that $f(\overline{A}) \subseteq \overline{f(A)}$. Let $B \subseteq Y$ be closed, and let $A = f^{-1}(B)$—we wish to show A is closed, and we do this by showing $A = \overline{A}$. Trivially, we note $A \subseteq \overline{A}$, so we show $\overline{A} \subseteq A$. As such, let $x \in \overline{A}$. Then note

$$f(x) \in f(\overline{A}) \subseteq \overline{f(A)} = \overline{f(f^{-1}(B))} \subseteq \overline{B} = B.$$

Thus $x \in f^{-1}(B) = A$, meaning $A = \overline{A}$, meaning $f^{-1}(B)$ is closed.

(3) $\Longrightarrow$ (1): Suppose that, for every closed set $B \subseteq Y$, the preimage $f^{-1}(B) \subseteq X$ is closed as well. Let $O \subseteq Y$ be open. Then note $Y \setminus O$ is closed in Y, and so $f^{-1}(Y \setminus O)$ is closed in X. Thus $X \setminus f^{-1}(Y \setminus O) = f^{-1}(O)$ is open, meaning f is continuous.

(1) $\Longrightarrow$ (4): Suppose f is continuous. Fix $x \in X$, then let $V \subseteq Y$ be an open neighborhood of $f(x)$. Since f is continuous, $U = f^{-1}(V) \subseteq X$ is an open neighborhood of x. Then note $f(U) = f(f^{-1}(V)) \subseteq V$ as desired.

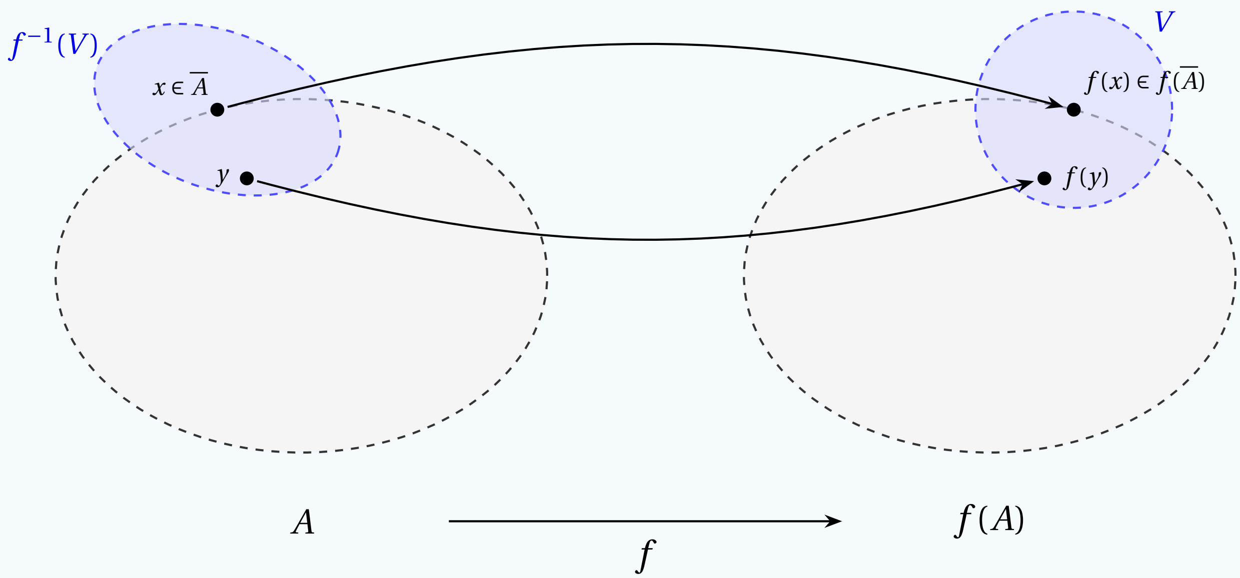
(4) $\Longrightarrow$ (1): Let $V \subseteq Y$ be an arbitrary open set. We seek to show $f^{-1}(V) \subseteq X$ is open. If $f^{-1}(V) = \emptyset$, then we are done. Otherwise, let $x \in f^{-1}(V)$. Then $f(x) \in V$, but there is some open neighborhood $U_x \subseteq X$ of x such that $f(U_x) \subseteq V$. Note then that $U_x \subseteq f^{-1}(V)$ for each $x \in f^{-1}(V)$, and so $\bigcup_{x \in f^{-1}(V)} U_x \subseteq f^{-1}(V)$. Moreover, $\{U_x\}_{x \in f^{-1}(V)}$ covers $f^{-1}(V)$, so $f^{-1}(V) \subseteq \bigcup_{x \in f^{-1}(V)} U_x$. Thus,

$$f^{-1}(V) = \bigcup_{x \in f^{-1}(V)} U_x,$$

which is the arbitrary union of open sets, completing the proof. ❦

[2.7.5] Remark (*Intuition Behind Characterizations of Continuity*)

The statement in (2), that $f(\overline{A}) \subseteq \overline{f(A)}$ for every $A \subseteq X$, can be interpreted locally and globally. In the case of metric spaces, if there is some $x \in \overline{A}$, then there is some sequence $(x_n) \subseteq A$ such that $x_n \to x$. If f is continuous, then the image sequence $f(x_n) \subseteq f(A)$ is such that $f(x_n) \to f(x)$. Thus elements of A and limit points of A are sent to elements of $f(A)$ and limit points of $f(A)$. That is, the image of the closure of A lives in the closure of $f(A)$, encoding the idea that $f(\overline{A}) \subseteq \overline{f(A)}$. This intuition locally reasons why sequential continuity works, and globally reasons that the continuous image of the closure of A lives in the closure of the continuous image of A. The global principle is illustrated in this schematic.



Statement (3) is a statement that is dual to the definition. Since open sets are dual to closed sets under complementation, it makes sense that changing open sets to closed ones in the definition of continuity yields the same characterization, since open subsets of a topological space are in bijective correspondence with closed subsets.

Statement (4) is a local characterization of continuity. In fact, we say a function f is continuous specifically at some point x_0 if, for every open neighborhood $V \subseteq Y$ of $f(x_0)$, there is some open neighborhood $U \subseteq X$ of x_0 such that $f(U) \subseteq V$. If this statement is true for all elements in the domain of f, then f is continuous on its domain. This statement is reminiscent of the challenge-response system used in $\varepsilon - \delta$ proofs in analysis. For a fixed element of the domain x and some neighborhood in the codomain about $f(x)$ (the "challenge"), we seek a neighborhood in the domain about x whose image lies in the neighborhood in the codomain (the "response").

[2.7.6] Definition (*Homeomorphism, Homeomorphic Spaces*)

Suppose X, Y are topological spaces and $f : X \to Y$ is a set-function. If f is bijective, and $f : X \to Y$ is continuous, and $f^{-1} : Y \to X$ is continuous, we say f is a **homeomorphism**. Two spaces X, Y are **homeomorphic** if a homeomorphism exists between them.

[2.7.7] Remark (*Equivalent Characterization of Homeomorphism*)

Recall $f : X \to Y$ is continuous if, for every open set $U \subseteq Y$, $f^{-1}(U) \subseteq X$ is open. Analogously, $f^{-1} : Y \to X$ is open if, for every open set $U \subseteq X$, $f(U) \subseteq Y$ is open. Thus a bijection $f : X \to Y$ such that $f(U)$ is open if and only if U is open is an alternate way to define a homeomorphism.

Note this means open sets between two homeomorphic spaces are in bijective correspondence. This property means homeomorphic spaces are not only identical as sets, but also identical as topological spaces! The structure of the two spaces are the same, and operations involving the topology of each space are analogous in each space. Homeomorphisms are akin to isomorphisms in that homeomorphic/isomorphic spaces are identical in structure and shape, up to labeling of elements.

[2.7.8] Definition (*Topological Embedding*)

Suppose $f : X \to Y$ is an injective, continuous map. Then let $\tilde{f} : X \to f(X)$ be the induced bijection from X to the image set $f(X)$. If $\tilde{f}$ is a homeomorphism, we say f is a **topological embedding** of X into Y. In other words, f is a topological embedding if it's an homeomorphism from its domain to its image.

We now list a few common types of continuous functions between topological spaces.

[2.7.9] Theorem (*Common Continuous Functions*)

Let X, Y, Z be topological spaces. Then the following functions are continuous.

(1) **Constant Function.** If $f : X \to Y$ maps every element of X to some $y_0 \in Y$, then f is continuous.

(2) **Inclusion Map.** If A is a subspace of X, the inclusion map $\iota : A \to X$ is continuous.

(3) **Composition.** If $f : X \to Y$ and $g : Y \to Z$ are both continuous functions, then $g \circ f : X \to Z$ is continuous.

(4) **Domain Restriction.** If $f : X \to Y$ is continuous and A is a subspace of X, then the restriction $f|_A : A \to Y$ is continuous.

(5) **Codomain Restriction/Expnasion.** If $f : X \to Y$ is continuous and Z is a subspace of Y that contains $f(X)$, then $g : X \to Z$ is continuous. Likewise, if Z is a superspace of Y, then $h : X \to Z$ is continuous as well.

(6) **Local Continuity.** The map $f : X \to Y$ is continuous if, for every $x \in X$, there is some neighborhood $U_x \subseteq X$ such that $f|_{U_x} : U_x \to Y$ is continuous.

[2.7.10] Theorem (*Pasting Lemma*)

Suppose $X = A \cup B$, where A, B are closed subsets of X. Let $f : A \to Y$ and $g : B \to Y$ be continuous functions such that $f(x) = g(x)$ for every $x \in A \cap B$. Define the function

$$h : X \to Y \quad h(x) = \begin{cases} f(x) & x \in A, \\ g(x) & x \in B. \end{cases}$$

Then h is continuous.

Proof. Let $O \subseteq Y$ be a closed subset of Y. Then note

$$h^{-1}(O) = f^{-1}(O) \cup g^{-1}(O).$$

By continuity, note $f^{-1}(O)$ is closed in A. By definition, this means $f^{-1}(O) = C \cap A$, where C is some closed subset of X. But since A is also a closed subset of X, this makes $f^{-1}(O)$ a closed subset of X. An entirely symmetric argument reveals $g^{-1}(O)$ is closed as well. Thus $h^{-1}(O)$ is closed in X, completing the proof. ❦

[2.7.11] Theorem (*Maps into Cartesian Product*)

Let $f : A \to X \times Y$ be a function defined by $f(x) = (f_X(x), f_Y(x))$. Then f is continuous if and only if $f_X : A \to X$ and $f_Y : A \to Y$ are both continuous.

Proof.

($\Longrightarrow$): Suppose f is continuous. Let $U \subseteq X$ be open: we seek to show $f_X^{-1}(U)$ is open as well. Note that $f_X = \pi_X \circ f$, so then $f_X^{-1} = (\pi_X \circ f)^{-1} = f^{-1} \circ (\pi_X)^{-1}$. Note that $\pi_X^{-1}(U) = U \times Y$, which is an open subset of $X \times Y$ when gifted with the product topology. Then, by continuity of f, the preimage of $U \times Y$ is open, meaning f_X^{-1} maps open sets to open sets, making f_X continuous. An entirely symmetric argument proves f_Y is continuous as desired.

($\Longleftarrow$): Suppose f_X, f_Y are continuous. Let $U \subseteq X \times Y$ be an open: we seek to show $f^{-1}(U)$ is open. Let $U = \bigcup_{\lambda \in \Lambda}(X_\lambda \times Y_\lambda)$, where $\{X_\lambda\}_{\lambda \in \Lambda}$ is a family of open subsets of X and $\{Y_\lambda\}_{\lambda \in \Lambda}$ is a family of open subsets in Y. Now note

$$\begin{aligned}
f^{-1}(U) &= \{x \in A : f(x) \in U\} \\
&= \left\{x \in A : f(x) \in \bigcup_{\lambda \in \Lambda}(X_\lambda \times Y_\lambda)\right\} \\
&= \bigcup_{\lambda \in \Lambda}\{x \in A : f_X(x) \in X_\lambda, f_Y(x) \in Y_\lambda\} \\
&= \bigcup_{\lambda \in \Lambda}\Big(\{x \in A : f_X(x) \in X_\lambda\} \cap \{x \in A : f_Y(x) \in Y_\lambda\}\Big) \\
&= \bigcup_{\lambda \in \Lambda}\Big(f_X^{-1}(X_\lambda) \cap f_Y^{-1}(Y_\lambda)\Big).
\end{aligned}$$

By continuity of f_X and f_Y, we note each $f_X^{-1}(X_\lambda) \cap f_Y^{-1}(Y_\lambda)$ is the intersection of two open sets, and is therefore open. Because $f^{-1}(U)$ is a union of such open sets, it is also open, meaning f is continuous. ❦

2.7.1 Exercises

[2.7.12] Problem

Suppose $f: X \to Y$ is continuous. Suppose x is a limit point of the subset $A \subseteq X$. Is it true that $f(x)$ is a limit point of $f(A)$?

Proof. Suppose x is a limit point of A. Let $V \subseteq Y$ be some open neighborhood of $f(x)$. By continuity of f, note there is some open neighborhood $U \subseteq X$ of x such that $f(U) \subseteq V$. Then note $U \cap (A \setminus \{x\}) \neq \emptyset$ by the definition of a limit point. If our scenario is such that $f(U \cap (A \setminus \{x\})) = \{f(x)\}$ for *every single* open neighborhood of x, then $f(x)$ is not a limit point of $f(A)$.

Let $U = X$. Then $f(U \cap (A \setminus \{x\})) = f(A \setminus \{x\}) = \{f(x)\}$ would make what we seek false. A candidate for disproof is a constant function. Let $f: \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = 0$, then let $A = [0,1] \subseteq \mathbb{R}$. Of course, 0 is a limit point for A, but $f(A) = \{0\}$, and singletons don't have limit points, completing the disproof. ❦

[2.7.13] Problem

Let $y_0 \in Y$. Define $f: X \to X \times Y$ by $f(x) = (x, y_0)$. Show that f is a topological embedding.

Proof. Note $f(X) = X \times \{y_0\}$, so we seek to show $\tilde{f}: X \to X \times \{y_0\}$ is a homeomorphism, where the codomain is blessed with the subspace topology. $\tilde{f}$ is obviously bijective, so we simply show $\tilde{f}$ and $\tilde{f}^{-1}$ are continuous.

First, suppose $V \subseteq X \times \{y_0\}$ is open. Then $V = V' \cap (X \times \{y_0\})$, where $V' \subseteq X \times Y$ is open. Note that

$$V' = \bigcup_{\lambda \in \Lambda} (X_\lambda \times Y_\lambda),$$

where $\{X_\lambda\}$ and $\{Y_\lambda\}$ are families of open subsets of X and Y respectively. Then

$$V = \left[\bigcup_{\lambda \in \Lambda} (X_\lambda \times Y_\lambda)\right] \cap \left[X \times \{y_0\}\right] = \underbrace{\left(\bigcup_{\lambda \in \Lambda, y_0 \in Y_\lambda} X_\lambda\right)}_{:= O_X} \times \{y_0\},$$

where $O_X \subseteq X$ is open. Then

$$\tilde{f}^{-1}(V) = \{x \in X : \tilde{f}(x) \in O_X \times \{y_0\}\} = \{x : x \in O_X\} = O_X,$$

an open subset of X.

Now suppose $U \subseteq X$ is open. Then

$$\tilde{f}(U) = U \times \{y_0\} = \underbrace{(U \times Y)}_{\text{open subset of } X \times Y} \cap (X \times \{y_0\}),$$

which is an open set in $X \times \{y_0\}$. Thus $\tilde{f}$ is a homeomorphism, making f an embedding. ❦

[2.7.14] Problem

Let Y be an ordered set endowed with the order topology. Let $f, g: X \to Y$ be continuous. Show the set $A = \{x : f(x) \leq g(x)\}$ is closed in X. Then, show $h(x) = \min\{f(x), g(x)\}$ is continuous.

Proof. We show $A = \overline{A}$. Let $x \in \overline{A}$ be a limit point of A. For the sake of contradiction, suppose $x \notin A$, and so $f(x) > g(x)$. If there is some $c \in Y$ such that $f(x) > c > g(x)$, then let $V_g = (-\infty, c)$ and $V_f = (c, \infty)$. Otherwise, let $V_f = (g(x), \infty)$ and $V_g = (-\infty, f(x))$. Then $V_f, V_g \subseteq Y$ are disjoint, open neighborhoods of $f(x)$ and $g(x)$, where all elements of V_f are strictly greater than all elements of V_g. By continuity, there exist open neighborhoods $U_f, U_g \subseteq X$ of x such that $f(U_f) \subseteq V_f$ and $g(U_g) \subseteq V_g$. But note $U_f \cap U_g$ is an open neighborhood of x. Since x is a limit point, this implies there is some $u \in (U_f \cap U_g) \cap (A \setminus \{x\})$. Since $u \in A$, it follows that $f(u) \le g(u)$. But also note $f(u) \in V_f$ and $g(u) \in V_g$, meaning $f(u) > g(u)$ by how V_f, V_g are defined. BY THE ***TRICHOTOMY POSTULATE OF ORDERED SETS***, this is a contradiction. Thus $f(x) \le g(x)$, and so $x \in A$, making A closed. An entirely symmetric argument shows $B = \{x : f(x) \ge g(x)\}$ is closed in X.

Define $\tilde{f} : A \to Y$ and $\tilde{g} : B \to Y$. By the pasting lemma, we may paste together these maps to form $h : A \cup B \to Y$ such that

$$h(x) = \begin{cases} f(x) & x \in A \\ g(x) & x \in B \end{cases} = \min\{f(x), g(x)\},$$

which is continuous as desired.

[2.7.15] Problem

Let $\{A_\alpha\}$ be a collection of subsets of X such that $X = \bigcup_\alpha A_\alpha$. Let $f : X \to Y$ be such that $f|_{A_\alpha}$ for each α.

(1) Supposing $\{A_\alpha\}$ is a finite family of closed sets, show f is continuous.

(2) Construct an example where $\{A_\alpha\}$ is a countable family of closed sets and f is discontinuous.

Proof.

- For our base case, suppose there are two sets in the family, A_1 and A_2. Note $f|_{A_1}$ and $f|_{A_2}$ are continuous and agree on mutual points in their domain, so gluing them together yields a continuous function, which is f. Now suppose the hypothesis is true for $n-1$ many sets. Let $A' = A_1 \cup A_2 \cup \cdots \cup A_{n-1}$, which is closed (since finite unions of closed sets are closed). $f|_{A'}$ is continuous by inductive hypothesis. Then note $f|_{A_n}$ is also continuous by the original hypothesis, and $f|_{A'}, f|_{A_n}$ agree on common points of their domain. Thus f is continuous by the Pasting Lemma.
- Let $X = [0,1]$. Let $\{A_n\}_{n \in \mathbb{Z}_{\ge 0}}$ be defined such that $A_0 = \{1\}$, $A_1 = \{0\}$, and $A_n = A_{n-1} \cup [0, 1 - 1/n]$ for every $n \ge 2 \in \mathbb{Z}_+$. Then each A_n is closed, since the finite union of closed intervals is closed, and $[0,1] = \bigcup_{n \in \mathbb{Z}_{\ge 0}} A_n$. Define $f : [0,1] \to \mathbb{R}$ by $f(x) = 0$ for all $x \in [0,1)$, and $f(1) = 1$ otherwise. Then note $f|_{A_n}$ is continuous for each $n \in \mathbb{Z}_+$, since f is constant on this restriction. But f is not continuous on $[0,1]$, since there is a jump discontinuity at $x = 1$.

[2.7.16] Problem

Let X be a topological space, and let $A \subseteq X$. Let $f : A \to Y$ be a continuous function to some target codomain Y that is Hausdorff. Show that if f can be uniquely extended to some continuous function $g : \overline{A} \to Y$, then g is uniquely determined by f.

Proof. Suppose $g_1, g_2 : \overline{A} \to Y$ are continuous functions such that $f = g_1|_A = g_2|_A$. For the sake of contradiction, suppose $g_1 \ne g_2$, so there is some $x \in \overline{A} \setminus A$ such that $g_1(x) \ne g_2(x)$. By Hausdorffness of Y, draw disjoint open neighborhoods $V_1, V_2 \subseteq Y$ of $g_1(x), g_2(x)$ respectively. By continuity of f, note $g_1^{-1}(V_1)$ and $g_2^{-1}(V_2)$ are open subsets of X that contain x. As such, $g_1^{-1}(V_1) \cap g_2^{-1}(V_2)$ is also an open neighborhood of x. Since x is a limit point of A, there is some $x_0 \in (g_1^{-1}(V_1) \cap g_2^{-1}(V_2)) \cap (A \setminus \{x\})$. Since $x_0 \in A$, it follows that $f(x_0) = g_1(x_0) = g_2(x_0)$. But $g_1(x_0) \in V_1$ and $g_2(x_0) \in V_2$, which is a contradiction. Thus $g_1 \equiv g_2$, completing the proof.

2.8 Box and Product Topology

We have a notion of Cartesian product for two sets X, Y. Namely, $X \times Y = \{(x, y) : x \in X, y \in Y\}$. But what about the Cartesian product of three sets? Four? Five? Countably many? To tackle this, we redefine the Cartesian product in terms of an arbitrary indexing set.

[2.8.1] Definition (*Cartesian Tuple, Cartesian Product*)

Let Λ be an indexing set. Given a set X, we define a **Λ-tuple** of X to be some function $\mathbf{x} : \Lambda \to X$. We define $x(\lambda)$, more commonly denoted as $\mathbf{x}_\lambda$, as the **λ-th component** of $\mathbf{x}$. We often write $(x_\lambda)_{\lambda\in\Lambda}$ in lieu of $\mathbf{x}$. We denote the set of all Λ-tuples on X by X^Λ, the set of all functions from Λ to X.

Now let $\{A_\lambda\}_{\lambda\in\Lambda}$ be an indexed family of set. We define the **Cartesian product** of this family of sets as the set of all Λ-tuples $(x_\lambda)_{\lambda\in\Lambda}$ such that each $x_\lambda \in A_\lambda$ (that is, the λ-th coordinate of $\mathbf{x}$ lies in A_λ). That is,

$$\text{Cartesian product of } \{A_\lambda\}_{\lambda\in\Lambda} = \prod_{\lambda\in\Lambda} A_\lambda = \left\{ x : \Lambda \to \bigcup_{\lambda\in\Lambda} A_\lambda : x_\lambda \in A_\lambda \, \forall \lambda \in \Lambda \right\}.$$

From here, we can extend the product topology of two sets in different ways. Recall how the basis and subbasis were defined for the product topology on $X \times Y$. We extend these notions to general indexing sets.

[2.8.2] Definition (*Box Topology, Product Topology*)

Suppose $\{X_\lambda\}_{\lambda\in\Lambda}$ is an indexed family of sets.

We define the **box topology** to be the topology generated by the basis

$$\mathcal{B} = \left\{ \prod_{\lambda\in\Lambda} U_\lambda : U_\lambda \text{ is open in } X_\lambda \right\}.$$

We define the **product topology** to be the topology generated by the subbasis

$$\mathcal{S}_\lambda = \{\pi_\lambda^{-1}(U_\lambda) : U_\lambda \text{ is open in } X_\lambda\} \qquad \mathcal{S} = \bigcup_{\lambda\in\Lambda} \mathcal{S}_\lambda.$$

Of course, if $\Lambda = \{1, 2\}$, then these are identical to our previous notion of the product topology on the Cartesian product of two sets. In fact, as we will realize soon, the box and product topology coincide for Cartesian products that come from finite indexing sets.

[2.8.3] Remark *(Comparison Between Box and Product Topology)*

We will compare these topologies by noticing the basis generated by the subbasis for the product topology, then comparing this basis with the basis for the box topology. From the previous definition, let $\{X_\lambda\}_{\lambda\in\Lambda}$ be a family of indexed sets, and define $\mathcal{S}_\lambda$ and $\mathcal{S}$ accordingly.

Recall that a basis is formed by finite intersection of elements of a subbasis. Let's first discover what the finite intersection of elements of $\mathcal{S}_\lambda$ do. Let $O_1,\dots,O_n$ be open subsets of X_λ, then note

$$\bigcap_{k=1}^{n} \pi_\lambda^{-1}(O_k) = \pi_\lambda^{-1}\underbrace{\left(\bigcap_{k=1}^{n} O_k\right)}_{\text{open in } X_\lambda},$$

which is another element of S_λ. Thus, a general basis element $B \in \mathcal{B}$ can be written as

$$B = \bigcap_{k=1}^{n} \pi_{\lambda_k}^{-1}(O_{\lambda_k}),$$

where $\pi_{\lambda_k} : \prod_{\lambda\in\Lambda} X_\lambda \to X_{\lambda_k}$ is the projection map on X_{λ_k}, and O_{λ_k} is an arbitrary open subset of X_{λ_k}. This implies an element $\mathbf{x} = (x_\lambda)_{\lambda\in\Lambda} \in \prod_{\lambda\in\Lambda} X_\lambda$ belongs to some basis B if $x_{\lambda_k} \in O_{\lambda_k}$ for each $k \in \{1,2,\dots,n\}$. Thus, we write B as a product

$$B = \prod_{\lambda\in\Lambda} U_\lambda,$$

where $U_\lambda \subseteq X_\lambda$ is open for $\lambda \in \{1,2,\dots,n\}$, and $U_\lambda = X_\lambda$ otherwise. Recalling that a basic open set in the box topology is the product of any open subset of X_λ for *each* $\lambda \in \Lambda$, not just finitely many of them, we conclude that the box topology is strictly finer than the product topology for an infinite indexing set, and they are otherwise equal for a finite one.

The implication of this is that, for infinite products, an element of a product belongs to a basic open set in the product topology if and only if finitely many of its coordinates lie in the finitely many open sets that generate the basis element. On the contrary, a basic open set in the box topology is determined by $|\Lambda|$ many open sets, not just finitely many.

The standard topology for the Cartesian product is the product topology.

[2.8.4] Theorem

Suppose $\{X_\lambda\}_{\lambda\in\Lambda}$ is an indexed family of sets, and let $X = \prod_{\lambda\in\Lambda} X_\lambda$. Then let $\{\mathcal{B}_\lambda\}_{\lambda\in\Lambda}$ be an indexed family of sets such that $\mathcal{B}_\lambda$ is a basis for each X_λ. Then

$$\mathcal{C} = \prod_{\lambda\in\Lambda} B_\lambda,$$

where $B_\lambda \in \mathcal{B}_\lambda$ for each $\lambda \in \Lambda$, forms a basis for the box topology on X. Alternatively, if $B_\lambda \in \mathcal{B}_\lambda$ for finitely many values of $\lambda \in \Lambda$, and otherwise $B_\lambda = X_\lambda$, then the product forms a basis for the product topology.

Proof. Let $X := \prod_{\lambda\in\Lambda} X_\lambda$ be endowed with the box topology, and let $O \subseteq X$ be open. We seek to show that, for each $x \in O$, there is some $C \in \mathcal{C}$ such that $x \in C \subseteq O$. If $x \in O$, then by the definition of the box topology, there exists a standard basic open set such that $x \in \prod_{\lambda\in\Lambda} O_\lambda \subseteq O$, where $O_\lambda \subseteq X_\lambda$ is open for each $\lambda \in \Lambda$.

But by definition, because $\mathcal{B}_\lambda$ is a basis for X_λ and the coordinate $x_\lambda \in O_\lambda$, we have that for each $\lambda \in \Lambda$, there exists a basis element $B_\lambda \in \mathcal{B}_\lambda$ such that

$$x_\lambda \in B_\lambda \subseteq O_\lambda.$$

But this implies there is a specific choice of sets to form $C = \prod_{\lambda\in\Lambda} B_\lambda \in \mathcal{C}$ such that

$$x \in \prod_{\lambda\in\Lambda} B_\lambda \subseteq \prod_{\lambda\in\Lambda} O_\lambda \subseteq O,$$

completing the proof that $\mathcal{C}$ is a basis for the box topology. An entirely analogous proof follows for the product topology, with the added condition that $O_\lambda = X_\lambda$ (and thus we can choose $B_\lambda = X_\lambda$) for all but finitely many values of λ. ❦

[2.8.5] Theorem (*Component-Wise Continuity*)

Suppose $\{X_\lambda\}_{\lambda\in\Lambda}$ is a family of indexed sets, where $X = \prod_{\lambda\in\Lambda} X_\lambda$. Suppose $f : A \to X$ is such that $f(a) = (f_\lambda(a))_{\lambda\in\Lambda}$ for every $a \in A$. Then f is continuous if and only if each f_λ is continuous.

Proof.

($\Longrightarrow$): Suppose f is continuous. For any $\lambda \in \Lambda$, note $f_\lambda = \pi_\lambda \circ f$. Note that the product topology forces π_λ to be continuous, f is continuous by hypothesis, and the composition of continuous functions is continuous, so f_λ is continuous.

($\Longleftarrow$): Suppose each f_λ is continuous. To show that f is continuous, it suffices to show that the preimage of every subbasis element of X is open in A. Let S be a standard subbasis element of the product topology on X. Then $S = \pi_\beta^{-1}(U_\beta)$ for some index $\beta \in \Lambda$ and some open set $U_\beta \subseteq X_\beta$. We examine the preimage of S under f:

$$f^{-1}(S) = f^{-1}(\pi_\beta^{-1}(U_\beta)) = (\pi_\beta \circ f)^{-1}(U_\beta) = f_\beta^{-1}(U_\beta).$$

Because f_β is continuous by hypothesis and U_β is open in X_β, the set $f_\beta^{-1}(U_\beta)$ is open in A. Thus, $f^{-1}(S)$ is open in A, meaning f is continuous. ❦

[2.8.6] Remark (*Component-Wise Continuity Fails for Box Topology*)

Define $\mathbb{R}^\omega$ to be the countably infinite product of $\mathbb{R}$ with itself. That is, $\mathbb{R}^\omega = \prod_{k\in\mathbb{Z}_+} \mathbb{R}$. Define the function $f : \mathbb{R} \to \mathbb{R}^\omega$ as the "identity" function $f(t) = (t, t, t, \ldots)$, where $\mathbb{R}$ takes the standard topology and $\mathbb{R}^\omega$ takes the box topology. Note then that $f(t) = (f_n(t))_{n\in\mathbb{Z}_+}$, where each $f_n(t) = t$. Clearly, each f_n is continuous. But is f continuous? Consider the basis element

$$B = (-1, 1) \times \left(-\frac{1}{2}, \frac{1}{2}\right) \times \left(-\frac{1}{3}, \frac{1}{3}\right) \times \cdots,$$

which is open in the box topology. If f is continuous, then $f^{-1}(B)$ must be open. But suppose $x \neq 0$ were in $f^{-1}(B)$—then $f(x) \in B$, meaning $f(x) = (x, x, x, \ldots) \in B$, but there is some $n \in \mathbb{Z}_+$ such that $x \notin (-1/n, 1/n)$, and so $f(x) \notin B$. 0, however, does lie in $f^{-1}(B)$, so this concludes $f^{-1}(B) = \{0\}$, which is not open in $\mathbb{R}$. Thus f is not continuous, emphasizing the previous result only holds for the product topology and *not* the box topology.

2.8.1 Exercises

[2.8.7] Problem

Let $\{X_\lambda\}$ be an indexed product of sets, and let $X = \prod_{\lambda\in\Lambda} X_\lambda$. Let $A_\lambda \subseteq X_\lambda$ be a subspace for each $\lambda \in \Lambda$, and let $A = \prod_{\lambda\in\Lambda} A_\lambda$. Show A is a subspace of X if both products are given the box topology or product topology.

Proof. To show $A \subseteq X$ is a subspace for X, we show that the basis for the box topology on A is the same as the subspace topology. Let B be an arbitrary basis element for the box topology on A, meaning $B = \prod_{\lambda\in\Lambda} O_\lambda$ where $O_\lambda \subseteq A_\lambda$ is open for each $\lambda \in \Lambda$. Since A_λ takes on the subspace topology, it follows that $O_\lambda = U_\lambda \cap A_\lambda$ for some open subset $U_\lambda \subseteq X_\lambda$. Then

$$B = \prod_{\lambda\in\Lambda} (U_\lambda \cap A_\lambda) = \left(\prod_{\lambda\in\Lambda} U_\lambda\right) \cap \left(\prod_{\lambda\in\Lambda} A_\lambda\right) = O \cap A,$$

where $O \subseteq X$ is open in the topology of X. Thus $O \cap A$ is open in the subspace topology of A. Conversely, we show that an arbitrary basis element–say, B—for the subspace topology on A is open in the box topology. Write $B = O \cap A$, where O is a basic open subset of X. Then $O = \prod_{\lambda\in\Lambda} O_\lambda$, where each $O_\lambda \subseteq X_\lambda$ is open, meaning we may write

$$B = \left(\prod_{\lambda\in\Lambda} O_\lambda\right) \cap \left(\prod_{\lambda\in\Lambda} A_\lambda\right) = \prod_{\lambda\in\Lambda} \underbrace{(O_\lambda \cap A_\lambda)}_{\text{open in } A_\lambda},$$

which is open in the box topology of A (by virtue of being a basis element). Thus the subspace and box topologies are equivalent for A, making A a subspace of X.

The proof for the product topology is entirely analogous. ❦

2.9 Metric Topology

[2.9.1] Definition (*Metric, Distance, ε-ball*)

A **metric** on some set X is a function $d : X \times X \to \mathbb{R}$ satisfying the following properties.

(1) **Nonnegativity.** $d(x, y) \geq 0$ for every $x, y \in X$.

(2) **Symmetry.** $d(x, y) = d(y, x)$ for every $x, y \in X$.

(3) **Triangle Inequality.** $d(x, y) + d(y, z) \geq d(x, z)$ for every $x, y, z \in X$.

We say $d(x, y)$ is the **distance** between two points $x, y \in X$. For some $\varepsilon > 0$ and $x_0 \in X$, we define an **ε-ball** about x_0 to be the set of all points a distance less than ε away from x_0. That is,

$$B_\varepsilon(x_0) = \{x \in X : d(x, x_0) < \varepsilon\}.$$

A topological space X is said to be **metrizable** if there exists some metric that induces the topology on X.

[2.9.2] Definition (*Metric Topology*)

Suppose (X, d) is a metric space. Then the set of all ε-balls about every point in X, given by

$$\mathcal{B} = \{B_\varepsilon(x) : \varepsilon > 0, x \in X\}$$

forms a basis for the **metric topology** on X.

Proof. We verify this forms a basis for X.

(1) **Covering.** For any $x \in X$, note $B_1(x)$ is in the basis, and so $X \subseteq \bigcup_{B \in \mathcal{B}} B$ as desired.

(2) **Local Refinement.** Let B_1, B_2 be arbitrary elements from the basis, and let $x \in B_1 \cap B_2$. Thus there must be some δ_1, δ_2 such that $B_{\delta_1}(x) \subseteq B_1$ and $B_{\delta_2}(x) \subseteq B_2$. Then note $B_{\min\{\delta_1, \delta_2\}}(x) \subseteq B_1 \cap B_2$, which is also a basic element, completing the proof.

❦

[2.9.3] Example (*Metric Topology Generates Discrete Topology*)

Given a set X, we may define a metric

$$d : X \times X \to \mathbb{R} \quad d(x, y) = \begin{cases} 1 & x \neq y, \\ 0 & \text{otherwise.} \end{cases}$$

It is obvious d is a metric. But note that the basis for the metric topology contains all singletons, since $\{x\} = B_1(x)$, so the metric topology is discrete.

[2.9.4] Example (*Metric Topology on $\mathbb{R}$ Generates Order Topology*)

Define the standard metric $d(x, y) = |x - y|$ on $\mathbb{R}$. Then note $(a, b) = B_\varepsilon(a + (b - a)/2)$, where $\varepsilon = (b - a)/2$. Similarly, all $B_\varepsilon(x) = (x - \varepsilon, x + \varepsilon)$. Thus the metric topology coincides with the standard order topology on $\mathbb{R}$.

[2.9.5] Definition (*Bounded Subset, Diameter*)

For some metric space (X,d), a set $A \subseteq X$ is said to be **bounded** if there is some $M \in \mathbb{R}$ such that $d(a_1,a_2) \le M$ for every pair $a_1,a_2 \in A$. The **diameter** of a bounded subset A is defined to be

$$\operatorname{diam}(A) = \sup\{d(a_1,a_2) : a_1,a_2 \in A\},$$

which exists since A is bounded above.

Topology is concerned with the idea of metrizability, the abstract notion that a metric may be endowed on a set to form a topology. The behavior and choice of a metric, however, is a strictly analytical topic. Thus, ideas such as "boundedness" that are dependent on choice of metric are not particularly topological. In fact, the next theorem shows that a different metric on $\mathbb{R}$ induces the same topology as the standard metric, making them indistinguishable topologically.

[2.9.6] Theorem (*Standard Bounded Metric Induces Same Topology as Standard Metric*)

Let (X,d) be a metric space. We define the **standard bounded metric** by

$$\overline{d} : X \times X \to \mathbb{R} \quad \overline{d}(x,y) = \min\{d(x,y),1\}.$$

Then $\overline{d}$ is a metric that induces the same metric topology as d.

Proof. The proof that $\overline{d}$ is a metric is trivial. Let $\mathcal{T}_d, \mathcal{T}_{\overline{d}}$ be the topologies generated by d and $\overline{d}$ respectively. It is obvious that $\mathcal{T}_{\overline{d}} \subseteq \mathcal{T}_d$, so we prove that $\mathcal{T}_d \subseteq \mathcal{T}_{\overline{d}}$. Let $B \in \mathcal{T}_d$. By definition of $\mathcal{T}_d$, for each $x \in B$, we can choose a basis ball about x contained in B; if its radius is 1 or greater, we can simply shrink it. So let B_x be a ball about x such that its radius is less than 1 and $x \in B_x \subseteq B$. Since its radius is less than 1, the ball B_x is identical under d and $\overline{d}$, meaning it is a basic open set of $\mathcal{T}_{\overline{d}}$. Then $B = \bigcup_{x \in B} B_x$, a union of basic open sets of $\mathcal{T}_{\overline{d}}$, completing the proof. ❦

[2.9.7] Example (*Metrizability of $\mathbb{R}^n$*)

Given $\mathbf{x} = (x_1,\ldots,x_n) \in \mathbb{R}^n$, we define $\|x\| = \sqrt{x_1^2 + \cdots + x_n^2}$ and define the **Euclidean metric**

$$d(x,y) = \|x - y\|,$$

the standard notion of distance between two points in $\mathbb{R}^n$. We can also define the **square metric** by

$$\rho(x,y) = \max\{|x_1 - y_1|, \ldots, |x_n - y_n|\}.$$

These are intuitive generalizations from the standard metric on $\mathbb{R}$. Indeed, the Euclidean and sqaure metric collapse to the standard metric in $\mathbb{R}^1$. Otherwise, consider basis elements for the Euclidean metric to be balls, whereas basis elements for the square metric are boxes.

[2.9.8] Theorem (*Criterion for Finer Metric Topologies*)

Let X be a set with two metrics d, d', and let $\mathcal{T}, \mathcal{T}'$ be the topologies induced by these metrics respectively. Then $\mathcal{T} \subseteq \mathcal{T}'$ if and only if, for each $x \in X$ and any $\varepsilon > 0$, there is some $\delta > 0$ such that $B_{d'}(x,\delta) \subseteq B_d(x,\varepsilon)$.

Proof.

($\Longrightarrow$): Suppose $\mathcal{T} \subseteq \mathcal{T}'$. Then, for any $x \in X$ and $\varepsilon > 0$, we have that $B_d(x,\varepsilon) \in \mathcal{T}$. Since $\mathcal{T} \subseteq \mathcal{T}'$, $B_d(x,\varepsilon) \in \mathcal{T}'$. Thus, for $x \in B_d(x,\varepsilon)$, by definition of the basis, there is some $\delta > 0$ such that $x \in B_{d'}(x,\delta) \subseteq B_d(x,\varepsilon)$.

($\Longleftarrow$): Let $O \in \mathcal{T}$—we seek to show $O \in \mathcal{T}'$. For each $x \in O$, note $x \in B_d(x,\varepsilon) \subseteq O$ for some $\varepsilon > 0$. By hypothesis, there is some $\delta > 0$ such that $B_x = B_{d'}(x,\delta)$ is such that $x \in B_x \subseteq B_d(x,\varepsilon) \subseteq O$. Then $O = \bigcup_{x \in O} B_x \in \mathcal{T}'$. ❦

[2.9.9] Theorem

The topologies on $\mathbb{R}^n$ induced by the Euclidean metric d and the square metric ρ are identical to the product topology on $\mathbb{R}^n$.

Proof. We begin by showing the Euclidean and square metrics generate the same topologies on $\mathbb{R}^n$. Recalling that basis elements for the Euclidean metric (resp. square metric) topology take the form of balls (resp. congruent boxes), we seek to show that a box can be contained in a sphere and vice versa. For starters, fix the points $x, y \in \mathbb{R}^n$, and let $\vec{y}$ denote the vector whose tip is located at x and tail is located at y. The magnitude of $\vec{y}$, representing the Euclidean distance $d(x, y)$, is bounded below by its largest single coordinate projection $\rho(x, y)$, meaning the box spanned by $\vec{y}$ easily fits within a sphere of radius $d(x, y)$. Conversely, if we bound $\vec{y}$ inside an n-dimensional cube centered at x whose faces are at a distance of $\rho(x, y)$, the Pythagorean theorem guarantees its magnitude cannot exceed the distance to the cube's furthest corner, yielding the upper bound $d(x, y) \leq \sqrt{n}\rho(x, y)$. The statement, which can also be proven with basic algebra, is that

$$\rho(x, y) \leq d(x, y) \leq \sqrt{n}\rho(x, y).$$

In any case, this implies $B_d(x, \varepsilon) \subseteq B_\rho(x, \varepsilon)$ for any $x \in X, \varepsilon > 0$, and same for $B_\rho(x, \varepsilon/\sqrt{n}) \subseteq B_d(x, \varepsilon)$. Thus the topologies are finer than eachother, making them equal.

It suffices to prove that the square metric generates the product topology. Let $\mathcal{T}_\rho$ be the topology generated by the square metric, and let $\mathcal{T}_\times$ be the product topology on $\mathbb{R}^n$. First, let $O \in \mathcal{T}_\rho$. For each $x \in O$, there exists some $\varepsilon > 0$ such that $x \in B_\rho(x, \varepsilon) \subseteq O$. But now note there is some box centered about x whose sidelengths are, at most, ε—this is spanned by basis elements of the product topology, and so $x \in$ (the box) $\subseteq B_\rho(x, \varepsilon)$. Thus $\mathcal{T}_\rho \subseteq \mathcal{T}_\times$.

Now suppose $O \in \mathcal{T}_\times$. For each $x \in O$, there is some $B = (a_1, b_1) \times \cdots \times (a_n, b_n)$ centered about x such that $x \in B \subseteq O$. With $\varepsilon = \min\{|a_i - b_i| : 1 \leq i \leq n\}$, we note $x \in B_\rho(x, \varepsilon/2) \subseteq B$, completing the proof. ❦

Now we seek to extend these metrices to an infinite product of $\mathbb{R}$. The natural generalizations on a set like $\mathbb{R}^\omega$ are as follows:

$$d(x, y) = \left[\sum_{i=1}^{\infty}(x_i - y_i)^2\right]^{1/2}$$

$$\rho(x, y) = \sup\{|x_n - y_n|\}.$$

But note $d(x, y)$ and $\rho(x, y)$ may not be well-defined due to convergence and boundedness issues. It is, however, possible to generalize a metric on $\mathbb{R}^\omega$, and on $\mathbb{R}^\Lambda$ for some general indexing set Λ. Recall the standard bounded metric $\overline{d}(x, y) = \min\{|x - y|, 1\}$—we may generalize it to $\mathbb{R}^\Lambda$ for some $(x_\lambda)_{\lambda\in\Lambda}, (y_\lambda)_{\lambda\in\Lambda}$ as such:

$$\overline{\rho}(x, y) = \sup_{\lambda\in\Lambda}\{\overline{d}(x_\lambda, y_\lambda)\}.$$

We refer to this as the **uniform metric** on $\mathbb{R}^\Lambda$, and it fittingly generates the **uniform topology**. This has implications for the idea of *uniform convergence*, which we will discuss soon.

[2.9.10] Theorem

For some indexing set Λ, the uniform topology on $\mathbb{R}^\Lambda$ is finer than the product topology and coarser than the box topology. That is, $\mathcal{T}_{\text{product}} \subseteq \mathcal{T}_{\text{uniform}} \subseteq \mathcal{T}_{\text{box}}$.

Proof. Let $\mathbf{x} = (x_\lambda)_{\lambda\in\Lambda} \in \mathbb{R}^\Lambda$. Let $U = \prod_{\lambda\in\Lambda} U_\lambda$ be an element of the product topology's basis about $\mathbf{x}$, where $\lambda_1, \ldots, \lambda_n$ are the indices for which $U_\lambda \neq \mathbb{R}$.

Let $z \in U$. We seek to show there is some $\varepsilon > 0$ such that $z \in B_{\overline{\rho}}(x, \varepsilon) \subseteq U$. For each λ_i, choose $\varepsilon_i > 0$ such that $B_{\overline{d}}(x, \varepsilon_i) \subseteq U_{\lambda_i}$. Then let $\varepsilon = \min\{\varepsilon_1, \ldots, \varepsilon_n\}$.

On the other hand, let $x \in B_{\overline{\rho}}(x, \varepsilon)$ for some $\varepsilon > 0$. For each $\lambda \in \Lambda$, let $X_\lambda = (x_\lambda - \varepsilon/2, x_\lambda + \varepsilon/2)$, then note $x \in \prod_{\lambda\in\Lambda} X_\lambda \subseteq B_{\overline{\rho}}(x, \varepsilon)$. ❦

[2.9.11] Theorem (*Continuity For Functions Between Metric Spaces*)

Let $f : X \to Y$ be a map between two metric spaces (X, d_X) and (Y, d_Y). Then f is continuous if and only if, for any $x \in X$ and $\varepsilon > 0$, there is some $\delta > 0$ such that

$$d_X(x, a) < \delta \implies d_Y(f(x), f(a)) < \varepsilon.$$

Proof.

($\Longrightarrow$): Suppose f is continuous. Fix $x \in X$ and $\varepsilon > 0$, and let $O = B_{d_Y}(f(x), \varepsilon)$. Since O is open, it follows that $f^{-1}(O)$ is open. Since $f^{-1}(O)$ is the union of basic, open sets in X, there is some $\delta > 0$ such that $B_{d_X}(x, \delta) \subseteq f^{-1}(O)$. Then, for any $a \in B_{d_X}(x, \delta)$, it follows that $f(a) \in O$.

($\Longleftarrow$): Suppose that, for any $x \in X$ and $\varepsilon > 0$, there is some $\delta > 0$ such that

$$d_X(x, a) < \delta \implies d_Y(f(x), f(a)) < \varepsilon.$$

Let O be open in Y—we show $f^{-1}(O)$ is open in X. Let $x \in f^{-1}(O)$. Let $\varepsilon > 0$ be such that $B_{d_Y}(f(x), \varepsilon) \subseteq O$. By hypothesis, there is some $\delta > 0$ such that $B_x = B_{d_X}(x, \delta)$, where $B_x \subseteq f^{-1}(O)$. Then $f^{-1}(O) = \bigcup_{x \in f^{-1}(O)} B_x$, which is open. ❦

[2.9.12] Theorem (*Sequence Lemma*)

Let X be a topological space, and let $A \subseteq X$. If there is a sequence $(x_n) \subseteq A$ such that $x_n \to x$, then $x \in \overline{A}$. Conversely, if X is metrizable and $x \in \overline{A}$, there is some sequence of points $(x_n) \subseteq A$ such that $x_n \to x$.

Proof.

($\Longrightarrow$): Suppose $(x_n) \subseteq A$ is such that $x_n \to x$. Then, for any open neighborhood O of x, there is some $N \in \mathbb{Z}_+$ such that $x_n \in O$ for every $n \geq N$. Thus $O \cap A \neq \emptyset$ and so $x \in \overline{A}$.

($\Longleftarrow$): Suppose X has some metric d. Fix $x \in \overline{A}$, and fix some arbitrary open neighborhood O of x. By definition of the closure, $O \cap A \neq \emptyset$. Since O is open in a metric space, there is some $\varepsilon > 0$ such that $B(x, \varepsilon) \subseteq O$ and $B(x, \varepsilon) \cap A \neq \emptyset$. Then there is some $N \in \mathbb{Z}_+$ such that $1/n < \varepsilon$ for every $n \geq N$. Let $x_n = a$ for some $a \in A$ and all $n < N$, and otherwise let x_n be some arbitrary element of $B(x, 1/n) \cap A$, which is guaranteed to exist. Then $(x_n) \subseteq A$ and $x_n \to x$. ❦

[2.9.13] Theorem (*Characterization of Continuity*)

Let $f : X \to Y$. If f is continuous, then for every $x_n \to x$ in X, it follows that $f(x_n) \to f(x)$. Conversely, if X is metrizable, we have that if every $x_n \to x$ in X is such that $f(x_n) \to f(x)$, then f is continuous.

Proof.

($\Longrightarrow$): Suppose f is continuous. Let $(x_n) \subseteq X$ be a sequence such that $x_n \to x$. Let O be an open neighborhood of $f(x)$. Then note $f^{-1}(O)$ is open by f's continuity, and $x \in f^{-1}(O)$. Thus there is some $N \in \mathbb{Z}_+$ such that $x_n \in f^{-1}(O)$ for every $n \geq N$. Thus $f(x_n) \in O$ for every $n \geq N$, meaning $f(x_n) \to f(x)$.

($\Longleftarrow$): Suppose X has some metric d. Suppose that, for every $(x_n) \subseteq X$ such that $x_n \to x$, it follows that $f(x_n) \to f(x)$. Let $A \subseteq X$—we will show f is continuous by proving $f(\overline{A}) \subseteq \overline{f(A)}$. For starters, suppose $x \in \overline{A}$. Since X is metrizable, A inherits a metric subspace topology, and by the sequence lemma there is some $(x_n) \subseteq A$ such that $x_n \to x$. By hypothesis, we have $f(x_n) \to f(x)$, meaning $f(x) \in \overline{f(A)}$. Thus $f(\overline{A}) \subseteq \overline{f(A)}$, completing the proof. ❦

[2.9.14] Definition (*Uniform Convergence*)

Let $f_n : X \to Y$ be a sequence of functions that map the set X into the metric space Y. We say $(f_n) \to f$ converges **uniformly** if, for any $\varepsilon > 0$, there is some $N \in \mathbb{Z}_+$ such that

$$d(f_n(x), f(x)) < \varepsilon$$

for every $n \geq N$ and every $x \in X$.

This mode of convergence is subtly distinct from traditional, pointwise convergence. Pointwise convergence only mandates that $(f_n(x)) \to f(x)$ for every $x \in X$ for large enough n, but this choice of "large enough" n can vary based on choice of x. If our choice of large enough n cannot be made independent from our choice of x, then the convergence is not uniform. Pointwise convergence is quite weak. We will soon see that if $(f_n) \to f$ with each f_n continuous, f is not necessarily continuous unless we impose extra hypotheses, which is where the uniformity of the convergence comes in.

[2.9.15] Theorem (*Uniform Limit Theorem*)

Suppose $f_n : X \to Y$ is a sequence of continuous functions that map the set X into the metric space Y. If $(f_n) \to f$ is uniform, then f is continuous.

Proof. Let $x \in X$ and $\varepsilon > 0$. We need to find $\delta > 0$ such that

$$d(x, y) < \delta \implies d(f(x), f(y)) < \varepsilon.$$

Since $f_n \to f$ is uniform, there is some $N \in \mathbb{Z}_+$ such that $d(f(x), f_N(x)) < \varepsilon/3$ for every $x \in X$. Then there is some $\delta > 0$ such that

$$d(x, y) < \delta \implies d(f_N(x), f_N(y)) < \varepsilon/3,$$

by continuity of each f_n. Then, if $d(x, y) < \delta$, we have that

$$\begin{aligned} d(f(x), f(y)) &\leq d(f(x), f_N(x)) + d(f_N(x), f_N(y)) + d(f_N(y), f(y)) \\ &= \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon, \end{aligned}$$

completing the proof. ❦

2.9.1 Exercises

[2.9.16] Problem

Let X be a set, and let $f_n : X \to \mathbb{R}$ be a sequence of functions. Let $\overline{\rho}$ be the uniform metric on the space $\mathbb{R}^X$, the set of all functions from X to $\mathbb{R}$. Show that $f_n \to f$ uniformly if and only if $f_n \to f$ as elements of the metric space $(\mathbb{R}^X, \overline{\rho})$.

Proof.

($\Longrightarrow$): Suppose $f_n \to f$ uniformly. To show $f_n \to f$ in $(\mathbb{R}^X, \overline{\rho})$, we choose some arbitrary open neighborhood O of f and show there is some $N \in \mathbb{Z}_+$ such that $f_n \in O$ for every $n \geq N$. Note there is some $\varepsilon > 0$ such that $f \in B(f, \varepsilon) \subseteq O$. By $f_n \to f$ being uniform, there is some $N_1 \in \mathbb{Z}_+$ such that $|f_n(x) - f(x)| < 1$ for every $n \geq N_1$. Then

$$\overline{\rho}(f_n, f) = \sup_{x \in X}\{\overline{d}(f_n(x), f(x))\} = \sup_{x \in X}\{|f_n(x) - f(x)|\},$$

with the second equality coming from the aforementioned fact. Again, by uniformity of $f_n \to f$, there is some $N > N_1 \in \mathbb{Z}_+$ such that $\overline{\rho}(f_n, f) \leq \varepsilon/2$ for every $n \geq N$. Thus $f_n \in B(f, \varepsilon) \subseteq O$ for every $n \geq N$, meaning $f_n \to f$ as elements of $\mathbb{R}^X$ as desired.

($\Longleftarrow$): Suppose $f_n \to f$ as elements of $\mathbb{R}^X$. Fix $\varepsilon > 0$. We seek to show there is some $N \in \mathbb{Z}_+$ such that

$$d(f_n(x), f(x)) < \varepsilon$$

for every $x \in X$ and $n \geq N$. Let $\varepsilon' = \min\{\varepsilon/2, 0.5\}$. Then let $O = B(f, \varepsilon')$ be an open set in $\mathbb{R}^X$. There is some $N \in \mathbb{Z}_+$ such that $f_n \in O$ for each $n \geq N$ by hypothesis. For any $n \geq N$, note that $\overline{\rho}(f_n, f) < \varepsilon' < 1$. Thus $\overline{d}(f_n(x), f(x)) < \varepsilon' < 1$ for every $x \in X$, and so $\overline{d}(f_n(x), f(x)) = d(f_n(x), f(x)) < \varepsilon' < \varepsilon$ for every $x \in X$ and $n \geq N$, meaning $(f_n) \to f$ uniformly.

2.10 Quotient Topology

[2.10.1] Definition (*Quotient Map*)

Let X, Y be topological spaces, and let $p : X \to Y$ be surjective. Then p is a **quotient map** if every subset $U \subseteq Y$ is open if and only if $p^{-1}(U)$ is open.

[2.10.2] Remark (*Stronger than Continuity, Weaker than Homeomorphism*)

A quotient map must be surjective, because then $p^{-1}(\emptyset) = \emptyset$. If $p^{-1}(\emptyset)$ weren't empty, then we can force a non-open set to be the preimage of the open set, contradicting the definition.

Note this is stronger than continuity. p would be continuous if $p^{-1}(U) \subseteq X$ was open given that $U \subseteq Y$ is open, which is only the reverse direction for the definition of a quotient map. For p to be a quotient map, it must be continuous, surjective, and it must be true that if $p^{-1}(U) \subseteq X$ is open, then $U \subseteq Y$ is too.

Note that a quotient map does **not** necessarily take an open $O \subseteq X$ to an open $p(O) \subseteq Y$. By definition, we'd need $p^{-1}(p(O))$ to be open, but $p^{-1}(p(O)) \subseteq O$, and subsets of O need not be open. $p^{-1}(p(O)) = O$ is forced when p is injective, but this would make p a bijective map that is continuous both ways, making p a homeomorphism. Indeed, a quotient map is simply a less restrictive homeomorphism, one that gets rid of injectivity in the hypothesis.

[2.10.3] Definition (*Saturated Set*)

A set $C \subseteq X$ is ***saturated*** *with respect to* f if it has the property that if it intersects some fiber $f^{-1}(\{y\})$, then C contains the entire fiber. That is, $C \subseteq X$ is saturated if it is the union of fibers of f.

We can reformulate a quotient map to be a map that takes open (resp. closed) saturated subsets of X to open (resp. closed) subsets of Y.

[2.10.4] Definition (*Open Map, Closed Map*)

Let $f : X \to Y$ be a map between two topological spaces. f is an **open map** if, for every open $U \subseteq X$, $f(U)$ is open. Analogously, f is a **closed map** if, for every closed $U \subseteq X$, $f(U)$ is closed.

Immediately from definition, it follows that surjective, continuous maps that are either open or closed are quotient maps. For example, the projection map $\pi : X \times Y \to X$ is surjective, continuous, and maps open sets to open sets, making it a quotient map.

[2.10.5] Definition (*Quotient Topology*)

Let X be a topological space and let A be a set. Let $p : X \to A$ be a surjective map. Then there is exactly one topology, the **quotient topology**, that can be endowed on A to make p a quotient map.

[2.10.6] Remark (*Verification of Quotient Topology*)

The definition of a quotient map makes the quotient topology unique. Indeed, with our previous definition in mind, our topology on A must be such that U is in the topology if and only if $p^{-1}(U)$ is open in X. Of course, the sets $\emptyset$ and A are open in the quotient topology, since $p^{-1}(\emptyset) = \emptyset$ and $p^{-1}(A) = X$, which are open sets in X. The conditions on arbitrary unions and finite intersections come by how preimages work:

$$p^{-1}\left(\bigcup_{\lambda\in\Lambda} U_\lambda\right) = \bigcup_{\lambda\in\Lambda} p^{-1}(U_\lambda), \qquad \text{[each } U_\lambda \text{ open in the quotient topology]}$$

$$p^{-1}\left(\bigcap_{k=1}^{n} U_k\right) = \bigcap_{k=1}^{n} p^{-1}(U_k), \qquad \text{[each } U_k \text{ open in the quotient topology]}$$

and so the quotient topology is a valid topology. Moreover, it is uniquely determined by the topology on X, since the only subsets of A that are open are the ones such that $p^{-1}(U)$ is open. Adding an additional set to the quotient topology would make p no longer continuous, whereas omitting an existing set in the quotient topology would make p no longer a quotient map.

[2.10.7] Definition (*Quotient Space*)

Let X be a topological space, and let X^* be a partition of X (a set of disjoint subsets of X whose union is X). Let $p : X \to X^*$ be an identification map that takes elements of X to the subset they belong to in X^*. In the quotient topology induced by p, X^* is called a **quotient space** of X.

[2.10.8] Remark (*Interlude on Equivalence Classes*)

Recall the idea behind the canonical set decomposition in Set and its representation as the **First Isomorphism Theorem** in Grp.

For any set X and some map out of it, say $f : X \to Y$, the image of the map induces an equivalence relation that partitions X. Call this partition X^*. We define $\ker(f) = \sim$ to be such that $x_1 \sim x_2 \iff f(x_1) = f(x_2)$. Then two elements of X belong to the same subset of X^* if their image is the same under f. Essentially, X^* changes the notion of equivalence, where two points of X are indistinguishable in X^* if they are "equivalent" to eachother, where equivalence is defined by having the same image under f.

We can decompose the action of f into three steps. First, we can define a map $\pi : X \to X^*$ that sends an element in X to its equivalence class in X^*. Then, since X^* is defined specifically so that elements of its equivalence classes have the same image, we can define a map $\tilde{f} : X^* \to \operatorname{im}(f)$ that sends each equivalence class to its image. Finally, we can define an inclusion map $\iota : \operatorname{im}(f) \to Y$ that embeds the image of f into its codomain. This is represented by this commutative diagram.

$$\begin{tikzcd}
A \arrow[r, two heads, "\pi"'] \arrow[rrr, bend left, "f"] & A/\sim \arrow[r, "\tilde{f}"'] & \operatorname{im}(f) \arrow[r, hook, "i"'] & B
\end{tikzcd}$$

Notice that $\tilde{f}$ is a bijection. Equating points of X whose image is equal under f resolves the non-injectivity of f, since the many points that may map to an element in the image collapse to one point under $\sim$. Mapping into $\operatorname{im}(f)$ instead of Y automatically resolves the non-surjectivity of f. If f is either injective or surjective, then these processes collapse trivially—namely, if f is injective, then X^* is a set of singleton subsets of X, and if f is surjective, then $\operatorname{im}(f) = Y$.

[2.10.9] Remark (*Reformulation of Open Sets in Quotient Space*)

Let X be a topological space and let X^* be a partition of X that is a quotient space relative to the identification map $p : X \to X^*$. Recall a set U is open in the quotient topology endowed on X^* is $p^{-1}(U)$ is open. A subset $U \subseteq X^*$ is a set of equivalence classes of X, and its preimage under p is simply the union of each equivalence class. Thus U is open if the union of equivalence classes it contains is an open subset of X.

Now we observe how the quotient map behaves in the context of subspaces, products, and other constructions. First, we note that the restriction of a quotient map to a subspace is not necessarily a quotient map. We will consider extra hypotheses that do let the restriction of a quotient map be a quotient map.

[2.10.10] Theorem (*Restriction of Quotient Map*)

Suppose $p : X \to Y$ is a quotient map, and let $A \subseteq X$ be a subspace. Then let $p|_A = q : A \to p(A)$ be the restriction.

(1) If A is an open or closed subset of X, then q is a quotient map.

(2) If p is an open or closed map, then q is a quotient map.

3 Connectedness and Compactness

3.1 Connectedness

[3.1.1] Definition (*Separations and Connectedness*)

Let X be a topological space. A **separation** of X is a pair of nonempty, disjoint open subsets $U, V \subseteq X$ whose union is X. We say X is **connected** if there exists no separation of X.

[3.1.2] Remark (*Intuition Behind Separation*)

Intuitively, we could consider a topological space X to be disconnected if we can find two non-empty, disjoint subsets $A, B \subseteq X$ whose union is X, such that neither set contains a limit point of the other. The requirement that neither set possesses any of the other's limit points entails that A and B are topologically isolated from each other, incapable of touching even at their boundaries. This is the geometric idea behind a disconnection. If X cannot be partitioned into sets that strictly exclude each other's limit points, then no part of X can be cleanly detached from the rest, meaning the space is connected. The equivalence between this geometric idea and the framing of a disconnection in terms of open sets will be visited soon.

[3.1.3] Theorem

Separations are formed by clopen sets.

> **Proof.** Suppose X is a topological space, and suppose $A, B \subseteq X$ are disjoint, non-empty, open sets whose union is X. Note also that $B = X \setminus A$ and $A = X \setminus B$, which are closed sets, form a separation as well. Thus A, B are clopen. ❦

[3.1.4] Theorem

A topological space X is connected if and only if the only clopen subsets of X are $\emptyset$ and X.

> **Proof.**
>
> ($\Longrightarrow$): Suppose X is connected. Then let O be a clopen subset of X. Note that $O, X \setminus O$ are disjoint open subsets of X whose union is X. For X to be connected, it cannot be separated, meaning $O, X \setminus O$ cannot be non-empty. Thus the only clopen subsets of X are $\emptyset, X$.
>
> ($\Longleftarrow$): Suppose the only clopen subsets of X are $\emptyset, X$. By the previous theorem, there exists no separation of X, so X is connected. ❦

We reformulate the idea of connectedness in terms of limit points.

[3.1.5] Theorem

Let Y be a subspace of the topological space X. A separation of Y is defined to be a pair of disjoint, non-empty sets $A, B \subseteq Y$ such that neither contains the limit points of the other. Then Y is connected if no separation exists.

Proof. First, we show that disjoint, non-empty sets $A, B \subseteq Y$ such that neither contains the limit points of the other are open. Since $\mathrm{Cl}_Y(A) \cap B = A \cap \mathrm{Cl}_Y(B) = \emptyset$, and $\mathrm{Cl}_Y(O) = \mathrm{Cl}_X(O) \cap Y$ for a general $O \subseteq Y$, it follows that $\mathrm{Cl}_X(A) \cap B = A \cap \mathrm{Cl}_X(B) = \emptyset$. Thus $\mathrm{Cl}_X(A) \cap Y = A$ and $\mathrm{Cl}_Y(B) \cap Y = B$, meaning A, B are closed in Y. Thus A, B are open as well (since $B = A^c$ and $A = B^c$).

Suppose $A, B \subseteq Y$ are disjoint, non-empty open sets whose union is Y. Note that A, B are clopen, and so $A = \mathrm{Cl}_Y(A) = \mathrm{Cl}_X(A) \cap Y$, and the intersection of this with B is $\emptyset$. Analogously, $A \cap \mathrm{Cl}_Y(B) = \emptyset$ as desired. ❦

[3.1.6] Theorem

Suppose X is a topological space with some separation $C, D \subseteq X$. If a subspace $Y \subseteq X$ is connected, then Y lies entirely within either C or D.

Proof. Note separations are formed by open sets, so $C, D \subseteq X$ are open in X and consequently $C \cap Y, D \cap Y \subseteq Y$ are open. Then

$$(C \cap Y) \cap (C \cap D) = (C \cap D) \cap Y = \emptyset \cap Y = \emptyset,$$
$$(C \cap Y) \cup (C \cap D) = (C \cup D) \cap Y = X \cap Y = Y.$$

Y is connected and thus no separation for it exists, but $C \cap Y, D \cap Y$ would form a separation for Y if one is non-empty. Thus, $Y = C \cap Y$ or $Y = D \cap Y$, meaning Y is a subset of one of C, D. ❦

[3.1.7] Theorem (*Arbitrary Union of Connected Spaces are Connected*)

Suppose X is a topological space and $\{O_\lambda\}_{\lambda \in \Lambda}$ is a family of connected subsets of X, each containing some common point $p \in X$. Then $O = \bigcup_{\lambda \in \Lambda} O_\lambda$ is connected.

Proof. Suppose X is a topological space and $\{O_\lambda\}_{\lambda \in \Lambda}$ is a family of connected subsets of X, each containing some common point $p \in X$. Let $O = \bigcup_{\lambda \in \Lambda} O_\lambda$. For the sake of contradiction, suppose $A, B \subseteq O$ is a separation for O. Without loss of generality, suppose $p \in A$. Then note each O_λ is connected, and so $O_\lambda \subseteq A$ is forced ($O_\lambda \not\subseteq B$ since $p \notin B$). Thus $O \subseteq A$, meaning $B = \emptyset$, a contradiction. ❦

[3.1.8] Theorem (*Continuous Image of Connected Set is Connected*)

The continuous image of a connected set is connected.

Proof. Let $f : X \to Y$ be a continous map from a connected topological space X to some set Y. Note then that $\tilde{f} : X \to f(X)$ is continuous—we seek to show $f(X)$ is connected. For the sake of contradiction, suppose $A, B \subseteq f(X)$ forms a separation for $f(X)$. Note that

$$X = \tilde{f}^{-1}(f(X)) = \tilde{f}^{-1}(A \cup B) = \tilde{f}^{-1}(A) \cup \tilde{f}^{-1}(B),$$

where the last expression is the union of two open sets by continuity of $\tilde{f}$. Then this is a separation of X, but X is connected, which is a contradiction. ❦

An immediate corollary to this theorem is the Intermediate Value Theorem, provided we have proven connected subsets of $\mathbb{R}$ are equivalent to intervals. We prove this later.

[3.1.9] Theorem (*Finite Products of Connected Sets are Connected*)

A finite product of connected topological spaces is connected.

Proof. Let X, Y be connected topological spaces: we seek to show $X \times Y$ is connected as well. To start, fix $(a, b) \in X \times Y$. Then note

$$X \cong (X \times b) \quad Y \cong (a \times Y).$$

Since connectedness is a topological property, it is preserved under homeomorphism. Thus

$$T_{(a,b)} = (X \times b) \cup (a \times Y)$$

is a union of two connected subspaces of $X \times Y$ that contains the point (a, b), so it is also connected. Then note

$$X \times Y = \bigcup_{(a,b) \in X \times Y} T_{(a,b)},$$

and so $X \times Y$ is connected. We may generalize this to finitely many products by induction. ❦

We will prove later that $\mathbb{R}$ is connected, but it is pretty intuitive to understand why. We discuss another advantage of the product topology over the box topology, in the fact that the product topology preserves more topological properties than the box topology.

[3.1.10] Remark (*Connectedness of $\mathbb{R}^\omega$ Under Box and Product Topologies*)

Consider $\mathbb{R}^\omega$ endowed with the box topology. Then note the set of bounded sequences, say O, and unbounded sequences, say O^c, form a partition for $\mathbb{R}^\omega$. We simply show these sets are open in the box topology. For any $\mathbf{a} \in \mathbb{R}^\omega$, define

$$U_a = (a_1 - 1, a_1 + 1) \times (a_2 - 1, a_2 + 1) \times \cdots,$$

which is open in the box topology. Then note each $U_\mathbf{a} \subseteq O$ for a bounded sequence $\mathbf{a}$, and so $O = \bigcup_{\mathbf{a} \in O} U_\mathbf{a}$, and so O is open. O^c can be proven to be open similarly, so O, O^c form a separation for $\mathbb{R}^\omega$ under the box topology.

Now consider $\mathbb{R}^\omega$ endowed with the product topology. For each $n \in \mathbb{Z}_+$, define $\tilde{\mathbb{R}}^n$ as the subspace of $\mathbb{R}^\omega$ that is the set of all sequences $\mathbf{x} = (x_1, x_2, \ldots)$, where each $x_i = 0$ for every $i > n$. Of course, $\tilde{\mathbb{R}}^n \cong \mathbb{R}^n$ for every $n \in \mathbb{Z}_+$, so these subspaces are connected. We define $\mathbb{R}^\infty = \bigcup_{n \in \mathbb{Z}_+} \tilde{\mathbb{R}}^n$, which is connected since each space share the common zero sequence. From an intuitive standpoint, we recognize $\mathbb{R}^\infty$ as the set of all sequences whose tails zero out. We seek to show that $\mathbb{R}^\omega$, endowed with the product topology, is the topological closure for $\mathbb{R}^\infty$—since the closure of a connected set is connected, our work will be done. For any $\mathbf{a} \in \mathbb{R}^\omega$, we show that any open neighborhood about $\mathbf{a}$ intersects $\mathbb{R}^\infty$. Let O be a basic open set of $\mathbb{R}^\omega$ that contains $\mathbf{a}$. In the product topology, this means there is some $n \in \mathbb{Z}_+$ such that

$$O = \left(\prod_{k=1}^{n} (a_k - \varepsilon_k, a_k + \varepsilon_k) \right) \times \prod_{n \in \mathbb{Z}_+ \setminus \{1, \cdots, n\}} \mathbb{R}.$$

Thus $(a_1, a_2, \ldots, a_n, 0, \ldots) \in O$, so $O \cap \mathbb{R}^\infty \neq \emptyset$.

Even though $\mathbb{R}$ is connected, the box topology does not preserve connectedness when $\mathbb{R}$ is multiplied by itself countably many times, whereas the product topology does. In fact, we prove later that an arbitrary product of connected sets is connected under the product topology.

3.1.1 Exercises

[3.1.11] Problem

Let X be a topological space, and let $\{A_n\}_{n \in \mathbb{Z}_+}$ be a family of connected topological subsets of X such that $A_n \cap A_{n+1} \neq \emptyset$. Show $A = \bigcup_{n \in \mathbb{Z}_+} A_n$ is a connected subset of X.

> **Proof.** For the sake of contradiction, suppose $A = B \cup C$ is a separation. Recall that each A_n is a connected subset of X: if A_n were disconnected in A, then a separation of it would exist in A (and thus consequently X), so A_n is also a connected subset of A in its subspace topology. Without loss of generality, we can say $A_1 \subseteq B$. Then $A_2 \subseteq B$ or $A_2 \subseteq C$, but $A_1 \cap A_2 \neq \emptyset$ and so $A_2 \subseteq B$ is forced. By induction, it follows every $A_n \subseteq B$, and so $A \subseteq B$ and $C = \emptyset$, a contradiction.

[3.1.12] Problem

Let $A \subseteq X$. Suppose $C \subseteq X$ is a connected subspace of X that intersects A and $X \setminus A$. Show that $C \cap \partial A \neq \emptyset$.

> **Proof.** Note that since $C \cap A \neq \emptyset$ and $C \cap A^c \neq \emptyset$, it follows that $C \cap \overline{A} \neq \emptyset$ and $C \cap \overline{(X \setminus A)} \neq \emptyset$. Moreover, note
>
> $$C = C \cap X = C \cap (\overline{A} \cup \overline{(X \setminus A)}) = \underbrace{(C \cap \overline{A})}_{:=A} \cup \underbrace{(C \cap \overline{(X \setminus A)})}_{:=B}.$$
>
> If $A \cap B = \emptyset$, then each set contains neither of the other's limit points and so this would be a separation of C, which is impossible since C is connected. Thus there is some $p \in A \cap B$, implying $p \in C \cap \partial A$.

[3.1.13] Problem

Let $\{X_\lambda\}_{\lambda\in\Lambda}$ be a family of connected spaces, and let $X = \prod_{\lambda\in\Lambda} X_\lambda$. Fix $a = (a_\lambda)_{\lambda\in\Lambda}$.

(1) For any finite $K \subseteq \Lambda$, define X_K to be the set of all points $x = (x_\lambda)_{\lambda\in\Lambda}$ such that $x_\lambda = a_\lambda$ for every $\lambda \notin K$. Show X_K is connected.

(2) Show that $Y = \bigcup_{K\subseteq\Lambda, |K|<\infty} X_K$ is connected.

(3) Show that $X = \overline{Y}$ and conclude that X is connected.

This result shows that the arbitrary product of connected sets is connected as well.

Proof.

(1) Note that
$$X_K \cong \prod_{\lambda\in K} X_\lambda \times \prod_{\lambda\notin K}\{a_\lambda\} = C \times \prod_{\lambda\notin K}\{a_\lambda\},$$
where C is the product of finitely many connected sets. Moreover, note $\prod_{\lambda\notin K}\{a_\lambda\}$ is a singleton, which is trivially connected. Thus, the product of these two yields a connected subset of X, and since it is homeomorphic to X_K, we are done.

(2) Note each X_K, for finite K, is connected. Moreover, each shares the common point a, so their union is connected.

(3) Note that $Y \subseteq X$, so $\overline{Y} \subseteq X$ immediately. Thus we show every $X \subseteq \overline{Y}$. Let $x \in X$, and let O be a basic open neigbhorhood of x in the product topology of X. Then note $O = \prod_{\lambda\in\Lambda} U_\lambda$, where $U_\lambda = X_\lambda$ for all but finitely many valued of λ, which we say are $\lambda_1, \ldots, \lambda_n$. Thus there is some point in O with arbitrary terms for indices $\lambda_1, \ldots, \lambda_n$ (such that they satisfy the product topology condition) and a_λ for all other values of λ. This point also belongs to Y, so $O \cap Y \neq \emptyset$. Since this basic open set is arbitrary, it follows that $X \subseteq \overline{Y}$, completing the proof.

❦

3.2 Connected Subspaces of the Real Line

[3.2.1] Definition (*Linear Continuum*)

A totally ordered set X with more than one element is said to be a **linear continuum** if the following hold:

(1) Any nonempty subset of X that is bounded above has a supremum.

(2) If $x < y \in X$, there is some $z \in X$ such that $x < z < y$.

[3.2.2] Theorem (*Connectedness of Linear Continuums*)

If L is a linear continuum in the order topology, then L is connected, and so are intervals/rays in L.

Proof. Recall that a subset $Y \subseteq L$ is said to be **convex** if, for any $a < b \in Y$, the interval $[a, b]$ of Y is a subset of Y (that is, all elements of L in between a and b lie in L). Note L is a convex subset of itself, and the intervals and rays in L are exactly the convex subsets of L, so let Y be an arbitrary convex subset of L—we prove it is connected.

For the sake of contradiction, suppose $Y = A \cup B$ is a separation. Pick some $a \in A$ and $b \in B$ (without loss of generality, let $a < b$), then note $[a, b] \subseteq Y$ by convexity of Y. Since A, B form a cover for Y, note that

$$A_0 = A \cap [a, b] \quad B_0 = B \cap [a, b]$$

forms a cover for $[a, b]$. Trivially, it follows that A_0 and B_0 are disjoint and open in the subspace topology $[a, b] \subseteq Y$. Thus $[a, b] = A_0 \cup B_0$ is a separation. Now let $c = \sup(A_0)$, which is well-defined since A_0 is a nonempty, bounded subset of a linear continuum. Since $[a, b]$ contains its maximum element b, it follows that $c \leq b$ and so $c \in [a, b]$. We show $c \notin A_0 \cup B_0 = [a, b]$ to yield a contradiction.

First, suppose $c \in B_0$. Then $c \neq a$, since $A_0 \cap B_0 = \emptyset$. By virtue of B_0 being open in $[a, b]$, there is some $a < d < b$ such that $(d, c] \subseteq B_0$. Then note we have have $(d, b] = (d, c] \cup (c, b]$. Indeed, $(c, b] \cap A_0 = \emptyset$ since c is greater than every element in A_0. Moreover, $(d, c] \subseteq B_0$, and so $(d, c] \cap A_0 = \emptyset$ since $B_0 \cap A_0 = \emptyset$. Thus d is an upper bound for A that is less than c, a contradiction.

Now suppose $c \in A_0$. Then note $c \neq b \in B_0$, so $a \leq c < b$. Since $A_0 \subseteq [a, b]$ is open, there is some $e \in A_0$ such that $[c, e) \in A_0$. Then there is some $c < z < e$ such that $z \in A_0$, a contradiction since c is meant to be an upper bound.

Thus Y cannot be separated. ❦

Since $\mathbb{R}$ is a linear continuum, it follows that $\mathbb{R}$ and all intervals/rays of $\mathbb{R}$ are connected. An immediate corollary is the Intermediate Value Theorem.

[3.2.3] Theorem (*Intermediate Value Theorem*)

Suppose $f : X \to Y$ is a continuous map from a connected set X to some ordered set Y in its order topology. For any two points $a, b \in X$, and for any $r \in Y$ such that $f(a) < r < f(b)$, then there is some $c \in X$ such that $f(c) = r$.

Proof. Let $A = f(X) \cap (-\infty, r)$ and $B = f(X) \cap (r, \infty)$. Note that A, B are disjoint and non-empty, since $f(a) \in A$ and $f(b) \in B$. Each is open in $f(X)$, being the intersection of an open set in Y (a ray) with $f(X)$ itself. Thus, if there is no $c \in X$ such that $f(c) = r$, then $f(X) = A \cup B$ would constitute a separation, which is a contradiction since the continuous image of a connected set is connected. Thus such a c exists. ❦

[3.2.4] Definition (*path-connectedness*)

Given points $x, y \in X$, we say a **path** from x to y is some continuous function $f : [a,b] \to X$ (where $[a,b] \subseteq \mathbb{R}$) such that $f(a) = x$ and $f(b) = y$. A space X is said to be **path-connected** if every two points in X can be joined by a path.

[3.2.5] Theorem (*path-connectedness Implies Connectedness*)

Suppose X is path-connected. Then X is connected.

Proof. Suppose X is path-connected, and for the sake of contradiction, let $X = A \cup B$ constitute a separation. For some $x \in A$ and $y \in B$, let $f : [a,b] \to X$ be a path from x to y. Then $f([a,b])$ is a connected subset of X, so it belong entirely in either A or B, a contradiction since its image has elements in both sets.

3.2.1 Exercises

[3.2.6] Problem

(1) Show no two of $(0,1), (0,1], [0,1]$ are homeomorphic.

(2) Suppose there exist embeddings $f : X \to Y$ and $g : Y \to X$. Give an example of X, Y that are not homeomorphic.

Proof.

(1) Suppose $f : (0,1) \to (0,1]$ is a homeomorphism. Since f is surjective, there is some $c \in (0,1)$ such that $f(c) = 1$. Then define $\tilde{f}$ by a domain restriction $\tilde{f} : (0,1) \setminus \{c\} \to (0,1)$, which is also a homeomorphism. But then note that even though the domain of $\tilde{f}^{-1}$ is connected, its image is not, a contradiction. An analogous argument works for all other pairs.

(2) Consider $X = (0,1]$ and $Y = [0,1]$. Then let f be the identity and let $g(x) = 0.5x + 0.5$.

[3.2.7] Problem

Suppose X is an ordered set in the order topology. Show that if X is connected, then X is a linear continuum.

Proof. Fix $a < b \in X$. If there were no $c \in X$ such that $a < c < b$, then b is the immediate successor to a, and so $X = (-\infty, b) \cup (a, \infty)$ constitutes a separation for X, which is impossible—thus $c \in X$.

Now suppose $O \subseteq X$ is nonempty and bounded above—we show there exists a supremum for O. Let A be the set of all upper bounds on O, and for the sake of contradiction, suppose A has no least element. Thus, for any $a \in A$, there is some $a' \in A$ such that $a' < a$. Note then that A contains no limit points of A^c and vice versa, so $A \cup A^c = X$ is a separation, a contradiction.

[3.2.8] Problem

Let X, Y be ordered sets in the order topology. Show if $f : X \to Y$ is order-preserving and surjective, then f is a homeomorphism.

Proof. Immediately, f is injective, since if $a \neq b$, it follows that $f(a) \neq f(b)$ by hypothesis—thus f is a bijection. Let $B = (a, b)$ be a basic open set in X (that is, either an interval or ray). For any $x < y \in B$, it follows that $f(x) < f(y)$ by hypothesis. If there is some $c \in Y$ such that $f(x) < c < f(y)$, then by surjectivity there is some $r \in X$ such that $f(r) = c$. By order preserving properties of f, r must be such that $x < r < y$, and so $r \in B$. Thus f sends an interval/ray in X to an interval/ray in Y. Note that $x \in B$ if and only if $a < x < b$; coupling this with the convexity of $f(B)$, we have that $f(B) = (f(a), f(b))$, an open interval in Y. An analogous argument can be used for f^{-1} as well, which proves f is a homeomorphism.

[3.2.9] Problem

(1) Is a product of path-connected spaces necessarily path-connected?

(2) Suppose $A \subseteq X$ is such that A is path-connected. Is $\overline{A}$ path-connected?

(3) Suppose X is path-connected and $f : X \to Y$ is a continuous map. Is $f(X)$ path-connected?

(4) Suppose $\{X_\lambda\}_{\lambda \in \Lambda}$ is a set of path-connected subsapces of X. If $\bigcap_{\lambda \in \Lambda} X_\lambda \neq \emptyset$, is it true that $\bigcup_{\lambda \in \Lambda} X_\lambda$ is path-connected?

Proof.

(1) Let $\{X_\lambda\}_{\lambda \in \Lambda}$ be a set of path-connected topological spaces, and let $X = \prod_{\lambda \in \Lambda} X_\lambda$. Let $\mathbf{x} = (x_\lambda)_{\lambda \in \Lambda}, \mathbf{y} = (y_\lambda)_{\lambda \in \Lambda} \in X$ be points. Note that for each λ, there is a continuous path $\gamma_\lambda : [a, b] \to X_\lambda$ such that $\gamma_\lambda(a) = x_\lambda$ and $\gamma_\lambda(b) = y_\lambda$. Now we may define

$$\gamma : [a, b] \to X \qquad \gamma(t) = (\gamma_\lambda(t))_{\lambda \in \Lambda},$$

then we see $\gamma(a) = \mathbf{x}$ and $\gamma(b) = \mathbf{y}$. Moreover, a map into the product topology is continuous so long as the components are continuous. Thus the product of path-connected spaces is path-connected.

(2) No. Let A be the image of the topologist's sine curve.

(3) Let $x, y \in X$. Then $x, y \in f^{-1}(f(X)) \subseteq X$. By path-connectedness of X, there is some $\gamma : [a, b] \to f^{-1}(f(X))$ such that $\gamma(a) = x$ and $\gamma(b) = y$. Then consider $f \circ \gamma : [a, b] \to f(X)$. $f \circ \gamma$ is the composition of continuous functions, so it is continuous. Moreover, $(f \circ \gamma)(a) = f(x)$ and $(f \circ \gamma)(b) = f(y)$. Thus this is a continuous path from $f(x)$ to $f(y)$, and since these are arbitrary points of $f(X)$, we conclude $f(X)$ is path-connected.

(4) Suppose $x, y \in \bigcup_{\lambda \in \Lambda} X_\lambda$: more specifically, let λ_x and λ_y be such that $x \in X_{\lambda_x}$ and $y \in X_{\lambda_y}$. By hypothesis there is some $z \in \bigcap_{\lambda \in \Lambda} X_\lambda$, so z is in both subspaces. By path-connectedness of each space, note there exists paths from x to z and y to z in the respective spaces. Merging these paths gives a new path from x to y that is contained in the union of all the path-connected spaces, completing the proof.

[3.2.10] Problem

Suppose $U \subseteq \mathbb{R}^2$ is an open, connected subspace of $\mathbb{R}^2$. Show U is path-connected.

Proof. Fix $x_0 \in U$, and let $O \subseteq U$ be the set of all points $x \in U$ such that there is a path from x to x_0. Let $x \in O$ be arbitrary: since $x \in U$ as well, there is some open ball B such that $x \in B \subseteq U$. For every $y \in B$, note there is a path from y to x (since balls in $\mathbb{R}^2$ are path-connected), and there is a path from x to x_0, so there is a path from y to x_0. Thus $y \in O$, and since y is arbitrary, this implies $B \subseteq O$. Since there is a open set about each point of O contained in O, we have that O is open.

Now consider O^c. Once again, let $x \in O^c$ be arbitrary, then there is some open ball B such that $x \in B \subseteq U$. For the sake of contradiction, suppose there is some $y \in B \cap O$. Then there is a path connection $x_0 \to y \to x$, which is a contradiction, since $x \in O$ in this case. Thus $B \subseteq O^c$ and so O^c is open.

Finally, note that O^c are disjoint, open sets such that $U = O \cup O^c$. Since U is connected, no separation exists, so one of these sets must be empty. Certainly, $x_0 \in O$, so O^c is empty and thus $O = U$, meaning U is path-connected.

3.3 Components and Local Connectedness

This section seeks to formalize the idea that, for any topological space X, there is a cover of connected and path-connected subsets of X.

[3.3.1] Definition (*Connected Components*)

Suppose X is a topological space. Define $\sim$ to be an equivalence relation on X such that, if $x \sim y$, then there is some connected subspace $C \subseteq X$ such that $x, y \in C$. The equivalence classes for $\sim$ are called the **connected components** of X.

Proof.

(1) **Reflexivity.** $x \sim x$ since $\{x\} \subseteq X$ is a connected subspace.

(2) **Symmetry.** Immediate.

(3) **Transitivity.** If $x \sim y$ and $y \sim z$, then there are connected subsets $A, B \subseteq X$ where $x, y \in A$ and $y, z \in B$. Then $A \cup B$ is a union that shares the point y, so it is connected, and it contains x, z. Thus $x \sim z$.

❦

[3.3.2] Theorem

The connected components of X are connected, disjoint subspaces of X whose union is X. Every nonempty connected subspace of X is fully contained in one equivalence class (i.e., it intersects exactly one equivalence class).

Proof. Since the connected components of X are equivalence classes, they form a partition for X. If $C \subseteq X$ were connected and were such that $C \cap [x_1]_\sim \neq \emptyset$ and $C \cap [x_2]_\sim \neq \emptyset$, where $\sim$ is the equivalence relation that induces the components, then $x_1 \sim x_2$ and so $[x_1]_\sim = [x_2]_\sim$—thus C intersects exactly one equivalence class and is also fully contained in one.

Fix some $x_0 \in X$, and note $[x_0]_\sim$ is a component. For any $x \in [x_0]_\sim$, there is some connected subspace C_x for which $x, x_0 \in C_x$. With the previous part in mind, note $C_x \subseteq [x_0]_\sim$, and so $[x_0]_\sim = \bigcup_{x \in [x_0]_\sim} C_x$, which is connected since each C_x has the shared point x_0. Thus every component is connected. ❦

[3.3.3] Remark

In a topological space, a component is referred to as a "maximal" connected subspace because it is completely saturated: any connected subspace belongs to exactly one component, and adjoining any outside element to a component would strictly break its connectedness.

An analogous process may be applied to split a space into path components.

[3.3.4] Definition (*Path Components*)

Suppose X is a topological space. Define $\sim$ to be an equivalence relation on X such that, if $x \sim y$, then there is some path from x to y contained in X. The equivalence classes for $\sim$ are called the **path components** of X.

Proof.

(1) **Reflexivity.** $x \sim x$ since $\gamma : [a, a] \to X$, where $\gamma(a) = x$, is a path from x to x.

(2) **Symmetry.** Immediate.

(3) **Transitivity.** Suppose $x \sim y$ and $y \sim z$. Then there are continuous maps $\gamma_1 : [a, b] \to X$ and $\gamma_2 : [b, c] \to X$ such that γ_1 parameterizes a path from x to y and γ_2 parameterizes a path from y to z. By the pasting lemma, note that $\gamma : [a, c] \to X$ defined by "pasting" γ_1, γ_2 is a path from x to z, and so $x \sim z$.

❦

[3.3.5] Theorem

The path components of X are path-connected, disjoint subspaces of X whose union is X. Every nonempty path-connected subspace of X is fully contained in one equivalence class (i.e., it intersects exactly one equivalence class).

Proof. The proof is identical to the proof about connected components. The proof for showing path components are path-connected is an immediate application of transitivity. ❦

[3.3.6] Remark

Note that the connected components C of a topological space X are closed. Note that if a subspace $A \subseteq X$ is connected, then so is its closure $\overline{A}$. Since all connected subspaces must be the subset of a connected component C, and the closure $\overline{C}$ of a connected component is connected, a connected component C equals its closure $\overline{C}$. Put another way, we established that connected components C are maximal, and so they must equal their closures. If there are finitely many connected components $C_1, \ldots, C_n$ of X, then each C_i is open, since the complement of one connected component C_i is the finite union of closed sets $\bigcup_{j \neq i} C_j$, which is closed.

To summarize, the connected components of a topological space are all closed, and if there are finitely many of them, they're all open.

[3.3.7] Definition (*Locally Connected, Locally Path Connected*)

Suppose X is a topological space. We say X is **locally connected at** x (resp. **locally path-connected at** x) if, for every neighborhood $U \subseteq X$ of x, there is some open, connected (resp. path connected) neighborhood $V \subseteq X$ of x fully contained in U. If all points in X are locally connected (resp. locally path connected), then we say X is locally connected (resp. locally path connected).

[3.3.8] Theorem

A space X is locally connected if and only if, for every open $U \subseteq X$, the connected components of U are open.

Proof.

($\Longrightarrow$): Suppose X is locally connected. Let $U \subseteq X$ be open, and let $C \subseteq U$ be some connected component of U. For each $x \in C$, there is some open, connected neighborhood V_x of x such that $x \in V_x \subseteq U$. Since $V_x \cap C \neq \emptyset$ and C is a connected component, it follows that $V_x \subseteq C$ since V_x is connected too. Thus $C = \bigcup_{x \in C} V_x$, the arbitrary union of open sets, which is open.

($\Longleftarrow$): Let $U \subseteq X$ be an arbitrary open subset. Then note, for each $x \in U$, there is some connected component $C \subseteq U$ such that $x \in C \subseteq U$. By hypothesis, C is open, completing the proof. ❦

[3.3.9] Theorem

A space X is locally path-connected if, for each open $U \subseteq X$, the path components of U are open.

> **Proof.** The proof is identical to the proof of the previous theorem. ❦

[3.3.10] Theorem

If X is a topological space, then each path component of X is fully contained in one connected component of X. If X is locally path-connected, then the path components and connected components coincide.

Proof. Let P be a path component of X. Since P is path-connected, it is also connected, so it is fully contained in some connected component $C \subseteq X$, proving the first statement.

Now assume X is locally path-connected. To prove the second statement, we show $C \subseteq P$. For the sake of contradiction, suppose $P \subsetneq C$. Let Q be the union of all path components of X that are different from P and intersect C. Since all path components are contained in exactly one connected component, it follows that $Q \subseteq C$. Then note

$$C = C \cap X = C \cap (P \cup P^c) = (C \cap P) \cup (C \cap P^c) = P \cup Q.$$

Since every path component of a locally path-connected set is open, we have that P, Q are disjoint, nonempty open subsets whose union is C, meaning C is separated (a contradiction, since C is connected). Thus $C \subseteq P$, and so $P = C$. ❦

3.3.1 Exercises

[3.3.11] Problem

What are the connected components and path components of $\mathbb{R}_\ell$? What are the continuous maps $f : \mathbb{R} \to \mathbb{R}_\ell$?

Proof. Suppose $C \subseteq \mathbb{R}_\ell$ contains at least two distinct points, $x < y$. Then $C = \big((-\infty, y) \cap C\big) \cup \big([y, \infty) \cap C\big)$ constitutes a separation of C, so C is disconnected. Thus, the only nonempty connected subspaces of $\mathbb{R}_\ell$ are singletons, which are therefore the connected components. Since every path-connected space is connected, the path components must also be exactly the singletons.

Note that for some $f : \mathbb{R} \to \mathbb{R}_\ell$ to be continuous, the image of a connected set must be connected. Since $\mathbb{R}$ is connected in its standard topology, it follows that $f(\mathbb{R})$ is connected, but the only connected subsets of $\mathbb{R}_\ell$ are singletons, so it follows that $f(\mathbb{R}) = \{*\}$, so all continuous maps are constant maps. ❦

[3.3.12] Problem

(1) What are the connected components and path components of $\mathbb{R}^\omega$ in the product topology?

(2) Consider $\mathbb{R}^\omega$ in the uniform topology. Show that $\mathbf{x},\mathbf{y}$ lie in the same component of $\mathbb{R}^\omega$ if and only if the sequence $(\mathbf{x}-\mathbf{y})$ is bounded above.

Proof.

(1) Note that the arbitrary product of connected and path-connected sets is connected and path-connected in the product topology. Thus $\mathbb{R}^\omega$ is the only connected and path-connected component in $\mathbb{R}^\omega$ with the product topology.

(2) $(\Longrightarrow)$: Suppose $\mathbf{x},\mathbf{y}$ are in the same connected component. Then there is some connected $C' \subseteq \mathbb{R}^\omega$ such that $\mathbf{x},\mathbf{y} \in C'$. Now consider a translation $T : \mathbb{R}^\omega \to \mathbb{R}^\omega$ defined by $T(\mathbf{x}) = \mathbf{x}-\mathbf{y}$. This is a homeomorphism, so $C := T(C')$ is also a connected component. Thus we have $\mathbf{x}-\mathbf{y}, \mathbf{0} \in C$.

First, we show $\mathbb{R}^\omega$ can be separated by the subspaces of bounded and unbounded sequences. Let $O \subseteq \mathbb{R}^\omega$ be the subspace of all bounded sequences. For each bounded sequence $x \in O$, define $B_x = B_{\overline{\rho}}(x, 1/2)$, a ball about x defined by the uniform metric of radius $1/2$. For each $y \in B_x$, note that $|x_n - y_n| < 1/2$ for every $n \in \mathbb{Z}_+$, and so y is bounded as well, meaning $y \in O$. Thus $x \in B_x \subseteq O$, so $O = \bigcup_{x \in O} B_x$. An analogous procedure can be performed to show O^c is open, since drawing a ball of radius $1/2$ about an unbounded sequence still lies in O^c. Thus $\mathbb{R}^\omega = O \cup O^c$, a separation.

We know $\mathbf{0}$ is bounded. If $\mathbf{x}-\mathbf{y}$ were unbounded, then $C = (O \cap C) \cup (O^c \cap C)$ would be a valid separation for a supposedly connected set (a contradiction), so $\mathbf{x}-\mathbf{y}$ is forced to be bounded.

$(\Longleftarrow)$: Suppose $\mathbf{x}-\mathbf{y}$ is a bounded sequence. Let $O \subseteq \mathbb{R}^\omega$ be the subspace of all bounded sequences. We will show O is connected, which will complete the proof.

We proceed by showing O is path-connected. Note the translation $T : \mathbb{R}^\omega \to \mathbb{R}^\omega$ given by $T(\mathbf{z}) = \mathbf{z}-\mathbf{y}$ is a homeomorphism. Since $\mathbf{x}-\mathbf{y}$ and $\mathbf{y}-\mathbf{y} = \mathbf{0}$ are both bounded sequences, we have $T(\mathbf{x}), T(\mathbf{y}) \in O$. We can show there is a path from $\mathbf{h} = \mathbf{x}-\mathbf{y}$ to $\mathbf{0}$ in O. We may define a map $\gamma : [0,1] \to O$ by $\gamma(t) = t\mathbf{h}$. Indeed, $\gamma(0) = \mathbf{0}$ and $\gamma(1) = \mathbf{h}$, so to show γ is a path, we simply show γ is continuous.

Since $\mathbf{h} \in O$, there is some $M \in \mathbb{R}$ such that $|h_n| < M$ for every $n \in \mathbb{Z}_+$. Fix some $t_0 \in [0,1]$ and some $1 > \varepsilon > 0$. Let $\delta = \varepsilon/M$, then let $t \in [0,1]$ be such that $|t - t_0| < \delta$. Then

$$\begin{aligned}
\overline{\rho}(\gamma(t), \gamma(t_0)) &= \sup_{n \in \mathbb{Z}_+} \{\min(|th_n - t_0 h_n|, 1)\} \\
&= \sup_{n \in \mathbb{Z}_+} \{|t - t_0||h_n|\} \\
&< \sup_{n \in \mathbb{Z}_+} \{(\varepsilon/M)(M)\} \\
&= \sup_{n \in \mathbb{Z}_+} \{\varepsilon\} = \varepsilon.
\end{aligned}$$

If $\varepsilon \geq 1$, then δ may be anything since the uniform metric is at most 1. Thus γ is continuous, and we conclude O is path-connected, which implies O is also connected. Because $T(\mathbf{x})$ and $T(\mathbf{y})$ both belong to the connected subspace O, they are in the same component. Since T^{-1} is a homeomorphism, it maps components to components, meaning $\mathbf{x}$ and $\mathbf{y}$ must also belong to the same component in $\mathbb{R}^\omega$, completing the proof.

[3.3.13] Problem

Let X be locally path-connected. Show all open, connected subsets of X are path-connected.

Proof. Let $O \subseteq X$ be open and connected. Fix $x \in O$ and any open neighborhood $U \subseteq O$ of x. Since U is an open subset of the locally path-connected space X, all of U's path components are open. Thus there is some open, path component (which is thus path-connected) $[x]_\sim$ such that $x \in [x]_\sim \subseteq U$. Thus O is locally path-connected, and so its path components and connected components coincide. Since O is connected, its only connected component is O, and so O is also path-connected, completing the proof. ❦

3.4 Compact Sets

[3.4.1] Definition (*Covering*)

A collection of subsets $\mathcal{A}$ of X is said to be a **cover** for X if the union of each set in $\mathcal{A}$ equals X. $\mathcal{A}$ is said to be an **open covering** if each element of $\mathcal{A}$ is an open subset of X.

Let $Y \subseteq X$. Then a collection of subsets $\mathcal{A}$ of X is said to be a cover for Y if Y is contained in the union of each set in $\mathcal{A}$.

[3.4.2] Definition (*Compact Set*)

Suppose X is a topological space. Then $C \subseteq X$ is said to be **compact** if every open cover of C has a finite subcover.

[3.4.3] Theorem (*Compactness of Subspace*)

Suppose X is a topological space, and $Y \subseteq X$ is a subspace. Then Y is compact if and only if, for every open cover of X that covers Y, there is a finite subcover that also covers Y.

Proof.

($\Longrightarrow$): Suppose $\mathcal{A}$ is a collection of open subsets of X that covers Y. Then note $\mathcal{A} \cap Y$, the intersection of each set in $\mathcal{A}$ by Y, forms an open cover of Y. Thus there is a finite subcover $\mathcal{B} \cap Y$ that covers Y. Then $\mathcal{B}$ is a finite collection of open subsets of X that covers Y.

($\Longleftarrow$): Let $\mathcal{B}$ be an open cover for Y. Each $B \in \mathcal{B}$ can be written as $B = A \cap Y$, where the set of all A's forms a collection of open subsets $\mathcal{A}$ of X that covers Y. There is a finite subcover of these that, when intersected with Y, forms a finite subcover for $\mathcal{B}$, completing the proof. ❦

[3.4.4] Theorem (*Closed Subspace of Compact Space is Compact*)

Any closed subspace of a compact space is compact.

Proof. Let X be a compact space, and let $Y \subseteq X$ be closed. Let $\mathcal{A}$ be an open cover of Y. Then note the collection defined by $\mathcal{B} = \mathcal{A} \cup \{X \setminus Y\}$ is an open cover of X ($X \setminus Y$ is open, since Y is closed). Since X is compact, there is a finite subcover B of X from $\mathcal{B}$. Then $B \setminus \{X \setminus Y\}$ is a finite subcover of $\mathcal{A}$, completing the proof. ❦

[3.4.5] Lemma (*Separation of a Point and a Compact Set*)

Suppose X is a Hausdorff space and $Y \subseteq X$ is compact. If $x_0 \notin Y$, then there are disjoint open neighborhoods containing x_0 and Y.

Proof. Fix $x_0 \in X \setminus Y$—we seek to find disjoint open neighborhoods containing x_0 and Y. For each $y \in Y$, it is obviously true that $x_0 \neq y$. Thus, by Hausdorffness of X, note that there are disjoint open neighborhoods $U_y \subseteq X$ of x_0 and $V_y \subseteq X$ of y. Then note $\{V_y\}_{y \in Y}$ forms a cover of Y by open sets from X, and by compactness of Y, we have that there exists a finite subcover $\{V_{y_k}\}_{k=1}^n$ of these open sets for Y. Thus we may write $Y \subseteq \bigcup_{k=1}^n V_{y_k}$.

Let $U = \bigcap_{k=1}^n U_{y_k}$ and $V = \bigcup_{k=1}^n V_{y_k}$, which are open subsets of X containing x_0 and Y, respectively. Then any $v \in V$ belongs to V_{y_k} for some $k \in \{1, 2, \dots, n\}$. By how V_{y_k} was chosen, it follows that $v \notin U_{y_k}$, and so $v \notin U$. Thus, U and V are disjoint open neighborhoods containing x_0 and Y. ❦

[3.4.6] Theorem (*Compact Subspace of Hausdorff Space is Closed*)

Any compact subspace of a Hausdorff space is closed.

Proof. Suppose X is a Hausdorff space, and let $Y \subseteq X$ be compact. We will show Y is closed by showing $X \setminus Y$ is open. Fix $x_0 \in X \setminus Y$. By the preceding lemma, note that there are disjoint open neighborhoods U and V containing x_0 and Y, respectively. Since $Y \subseteq V$ and U and V are disjoint, it follows that U cannot intersect Y. Thus $U \cap Y = \emptyset$, and so $U \subseteq X \setminus Y$. Thus $x_0 \in U \subseteq X \setminus Y$, implying $X \setminus Y$ is open and Y is closed. ❦

[3.4.7] Theorem (*Continuous Image of Compact Set is Compact*)

The continuous image of a compact set is compact.

Proof. Suppose X is a compact set, and $f : X \to Y$ is continuous. Let $\mathcal{A}$ be a cover for $f(X)$ with open subsets of Y. Since the preimage of an open set is open, the preimage $f^{-1}(\mathcal{A})$ is an open cover of X. By compactness, there is a finite subcover $\{f^{-1}(A_k)\}_{k=1}^n$ with open sets of X. Then $f\left(\{f^{-1}(A_k)\}_{k=1}^n\right) = \{A_k\}_{k=1}^n$ forms a finite subcover of $f(X)$, completing the proof. ❦

[3.4.8] Theorem

Suppose $f : X \to Y$ is a bijective, continuous function. If X is compact and Y is Hausdorff, then f is a homeomorphism.

Proof. To show f is a homeomorphism, we show f sends closed sets to closed sets. Let $O \subseteq X$ be closed, and thus consequently compact since all closed subsets of a compact space is compact. Thus $f(O) \subseteq Y$ is closed, and thus consequently closed since all compact subsets of a Hausdorff space are closed. Thus f sends closed sets to closed sets. ❦

[3.4.9] Lemma (*Tube Lemma*)

Suppose X, Y are topological spaces, with Y compact. Fix $x_0 \in X$. If $N \subseteq X \times Y$ is an open set containing $\{x_0\} \times Y$, then N contains some tube $W \times Y$ about $\{x_0\} \times Y$, where $W \subseteq X$ is an open neighborhood of x_0.

Proof. Fix $x_0 \in X$ and consider the set $\{x_0\} \times Y \subseteq X \times Y$. Let $N \subseteq X \times Y$ be an open neighborhood of $\{x_0\} \times Y$. For each $y \in Y$, choose some open neighborhoods U_y of x_0 and V_y of y such that $(x_0, y) \in U_y \times V_y \subseteq N$. Then $\{U_y \times V_y\}_{y \in Y}$ forms an open cover for $\{x_0\} \times Y$. Note that $\{x_0\} \times Y \cong Y$, and so the left-hand side is compact, and so there is a finite subcover $\{U_k \times V_k\}_{k=1}^n$ of $\{x_0\} \times Y$. Define

$$W = \bigcap_{k=1}^{n} U_k,$$

an open neighborhood in X about x_0. We will show $W \times Y \subseteq N$, completing the proof.

Let $(x, y) \in W \times Y$ be arbitrary. Then note $(x_0, y) \in \{x_0\} \times Y$, with y being the same as chosen from $W \times Y$. Thus there is some $j \in \{1, 2, \ldots, n\}$ such that $y \in V_j$, reason being that $\{U_k \times V_k\}_{k=1}^n$ is a finite subcover for $\{x_0\} \times Y$. Recall that $x \in U_j$ as well, since $x \in W$, and so $(x, y) \in (U_j \times V_j) \subseteq N$. Thus $W \times Y \subseteq N$, completing the proof. ❦

[3.4.10] Theorem (*Finite Product of Compact Spaces is Compact*)

The product of finitely many compact spaces is compact.

Proof. Suppose X, Y are compact spaces, and fix $x \in X$. Let $\mathcal{A}$ be an open cover for $X \times Y$—we show it has a finite subcover.

Consider the slice $\{x_0\} \times Y$: note that it is homeomorphic to the compact space Y, so there is a finite subcover $\{A_k\}_{k=1}^m$ for $\{x_0\} \times Y$. Let $N = \bigcup_{k=1}^m A_k$, an open set in $X \times Y$ containing $\{x_0\} \times Y$. By the Tube Lemma, there is some open neighborhood $W_x \subseteq X$ of x such that $W_x \times Y \subseteq N$. Thus $\{A_k\}_{k=1}^m$ forms a finite cover for $W_x \times Y$. Finally, note $\{W_x\}_{x \in X}$ is an open cover of X. By compactness of X, we have that $\{W_k\}_{k=1}^n$ forms a finite subcover for X. Thus we have that the union of the tubes

$$W_1 \times Y, W_2 \times Y, \ldots, W_n \times Y$$

is all of $X \times Y$; since each tube has a finite cover of open sets from $\mathcal{A}$, we have that $X \times Y$ is covered by finitely many sets from $\mathcal{A}$, completing the proof.

The proof for more compact sets proceeds by induction on the number of sets. ❦

We now investigate the finite intersection property and a reformulation of compact spaces in terms of them.

[3.4.11] Definition (*Finite Intersection Property*)

A collection of sets $\mathcal{C}$ is said to have the **finite intersection property** if, for every finite subcollection $\{C_k\}_{k=1}^n$ of $\mathcal{C}$, $\bigcap_{k=1}^n C_k \neq \emptyset$.

[3.4.12] Theorem (*Compactness in Terms of Finite Intersection Property*)

Let X be a topological space. Then X is compact if and only if, for every collection $\mathcal{C}$ of closed subsets of X that have the finite intersection property, the intersection $\bigcap_{C\in\mathcal{C}} C$ of all the elements of $\mathcal{C}$ is nonempty.

Proof. Given a collection $\mathcal{A}$ of subsets of X, we may define

$$\mathcal{C} = \{X \setminus A : A \in \mathcal{A}\}.$$

Then the following statements are true:

(1) $\mathcal{A}$ is a collection of open subsets of X if and only if $\mathcal{C}$ is a collection of closed subsets of X.

(2) $\mathcal{A}$ forms a cover for X if and only if $\bigcap_{C\in\mathcal{C}} C = \emptyset$.

(3) $\{A_k\}_{k=1}^n$ forms a finite subcover for X if and only if $\bigcap_{k=1}^n C_k = \emptyset$, where $C_k = X \setminus A_k$.

The first statement is by definition of a closed set, and the other two are a direct application of De Morgan's Laws. Namely that

$$X \setminus \left(\bigcup_{\lambda\in\Lambda} A_\lambda\right) = \bigcap_{\lambda\in\Lambda} (X \setminus A_\lambda).$$

Consider the original definition of compactness: X is compact if and only if, for any open cover $\mathcal{A}$, there is a finite subcover. Thus if $\mathcal{A}$ has no finite subcover for X, then it follows that $\mathcal{A}$ does not cover the compact set X. Now let $\mathcal{C} = \{X \setminus A : A \in \mathcal{A}\}$ as before. Immediately we see that compactness may be rephrased as such:

"*We say that X is compact if, for every collection of closed subsets $\mathcal{C}$ of X with the finite intersection property, the intersection of all elements in $\mathcal{C}$ is nonempty.*"

❦

[3.4.13] Remark (*Cantor's Intersection Theorem*)

Suppose $\{C_k\}_{k\in\mathbb{Z}_+}$ is a sequence of closed subsets of a compact space X, such that $C_k \supseteq C_{k+1}$ for each $k \in \mathbb{Z}_+$. Immediately we see that any finite subcollection of $\{C_k\}$ has nonempty intersection, meaning this sequence has the finite intersection property. By compactness, we have that $\bigcap_{C\in\mathcal{C}} C \neq \emptyset$. This result is familiar to us with $X = \mathbb{R}$.

[3.4.14] Recap

(1) For any topological space X, a collection of subsets $\mathcal{A}$ of X is said to **cover** X if $\bigcup_{A \in \mathcal{A}} A = X$. An **open cover** is a cover comprised of open sets. A subcover of X is a subset of $\mathcal{A}$ that still manages to cover X.

(2) A space X is said to be **compact** if, for every open cover $\mathcal{A}$, there is a finite subcover of X.

(3) For a subspace $Y \subseteq X$, a collection of subsets $\mathcal{A}$ of X is said to cover Y if $Y \subseteq \bigcup_{A \in \mathcal{A}} A$. This is different from the traditional definition of a cover for one space.

(4) A space $Y \subseteq X$ is compact if and only if, for every open cover of Y from open subsets of X, there is a finite subcover.

> **Intuition.** The idea behind this proof is that an open set in Y is an open set in X intersected with Y. Thus open covers can be moved around seamlessly between X and Y: simply move an open cover to the space that is given as compact by hypothesis, take the finite subcover, then move it back to the other space.

(5) Every closed subspace of a compact space is compact.

> **Intuition.** Let X be compact and $Y \subseteq X$ closed. Of course, $X \setminus Y$ is open. For any open cover of Y, we may adjoin $X \setminus Y$ to form an open cover for X, use its compactness, then simply get rid of the $X \setminus Y$ term to get a finite subcover of Y.

(6) Every compact subspace of a Hausdorff space is closed.

> **Intuition.** Let X be Hausdorff and $Y \subseteq X$ compact. We want to show $X \setminus Y$ is open. Fixing $x_0 \in X \setminus Y$, we show the set is open by showing there exists an open neighborhood $U \subseteq X$ of x_0 lying in $X \setminus Y$—that is, $U \cap Y = \varnothing$. For each $y \in Y$, we have that $y \neq x_0$, so by Hausdorffness there exist disjoint open neighborhoods $U_y \subseteq X$ of x_0 and $V_y \subseteq X$ of y. Then note $\{V_y\}_{y \in Y}$ forms an open cover for the compact space Y, so there exists a finite subcover $\{V_{y_k}\}_{k=1}^n$ of Y. Then define $U = \bigcap_{k=1}^n U_{y_k}$, an open neighborhood of x_0. For any $y \in Y$, there is some $k \in \{1, \ldots, n\}$ such that $y \in V_{y_k}$, implying $y \notin U_{y_k}$, so $y \notin U$.

(7) We extract the key mechanism from the last proof: for any compact subspace Y of a Hausdorff space X, we can always construct disjoint open neighborhoods U, V about x_0, Y respectively—that is, $x_0 \in U \subseteq X \setminus Y$, meaning $X \setminus Y$ is open.

(8) The continuous image of a compact set is compact.

> **Intuition.** Let $f : X \to Y$ be a continuous map from the compact space X to a target space Y. Let $\mathcal{A}$ be an open covering for $f(X)$. Then note $f^{-1}(\mathcal{A})$ is an open covering of X by continuity of f. Since X is compact, we have a finite subcover $\{f^{-1}(A_k)\}_{k=1}^n$. Pushing forward yields that $f(X)$ is covered by the finite subcover $\{A_k\}_{k=1}^n$.

(9) Let $f : X \to Y$ be a bijective, continuous function. If X is compact and Y is Hausdorff, then f is a homeomorphism.

> **Intuition.** We show f^{-1} is continuous by showing f takes closed sets to closed sets. For some closed set $C \subseteq X$, we note C is also compact by a previous theorem. Since f is continuous, it follows that $f(C) \subseteq Y$ is compact in a Hausdorff space, which makes $f(C)$ closed.

(10) Let X, Y be topological spaces, with Y compact. For fixed $x_0 \in X$ and any open neighborhood $N \subseteq X \times Y$ of $\{x_0\} \times Y$, there is some open neighborhood $W \subseteq X$ of x_0 such that $W \times Y \subseteq N$.

> **Intuition.** Fix $x_0 \in X$ and let $N \subseteq X \times Y$ be an open neighborhood of $\{x_0\} \times Y$. Form a cover for $\{x_0\} \times Y$, of the form $\{U_y \times V_y\}_{y \in Y}$, where $U_y \times V_y$ is a basic, open neighborhood about (x_0, y). Note that $\{x_0\} \times Y \cong Y$, and since Y is compact, we form an open cover $\{U_k \times V_k\}_{k=1}^n$. Then we define $W = \bigcap_{k=1}^n U_k$, an open neighborhood of x_0. We show $\{U_k \times V_k\}_{k=1}^n$ covers not only $\{x_0\} \times Y$, but also $W \times Y$. Let $(x, y) \in W \times Y$ be arbitrary, and let $(x_0, y) \in \{x_0\} \times Y$ be the corresponding element with the same y-coordinate. Thus $y \in V_j$ for some $j \in \{1, \ldots, n\}$, and since $x \in U_k$ for every k, $x \in U_j$. Thus $(x, y) \in (U_j \times V_j) \subseteq N$, completing the proof.

(11) The product of finitely many compact sets is compact.

> **Intuition.** We prove this for two compact topological spaces X, Y, and the full theorem is proven in a straightforward manner by induction. Suppose $\mathcal{A}$ is a cover of $X \times Y$. Fix $x \in X$, then note $\{x\} \times Y \cong Y$ is compact, so there is some finite subcover $\{A_k\}_{k=1}^m$ of sets from $\mathcal{A}$. Then $N = \bigcup_{k=1}^m A_k$ is an open subset of $X \times Y$ containing $\{x\} \times Y$. By the Tube Lemma, there is some open neighborhood $W_x \subseteq X$ of x such that $W_x \times Y \subseteq N$. Then $\{W_x\}_{x \in X}$ forms an open cover for X, and since X is compact, there is a finite subcover $\{W_k\}_{k=1}^n$. Thus
>
> $$\{W_1 \times Y, W_2 \times Y, \ldots, W_n \times Y\}$$
>
> forms an open cover for $X \times Y$. Each element of this set also has a finite subcover of elements from $\mathcal{A}$, thus the entirety of $X \times Y$ has a finite subcover from elements of $\mathcal{A}$.

(12) A topological space X is compact if, for every collection of closed subsets $\mathcal{C}$ with the finite intersection property, the intersection $\bigcap_{C \in \mathcal{C}} C \neq \emptyset$.

> **Intuition.** Observe that, for any collection of subsets $\{\mathcal{A}_\lambda\}_{\lambda \in \Lambda}$ of X, it is true that
>
> $$X \setminus \left(\bigcup_{\lambda \in \Lambda} A_\lambda \right) = \bigcap_{\lambda \in \Lambda} (X \setminus A_\lambda).$$
>
> Defining $\mathcal{C} = \{X \setminus A : A \in \mathcal{A}\}$, where $\mathcal{A}$ is an open cover of X, and reformulating compactness in terms of $\mathcal{C}$ and the relation above yields this theorem.

3.4.1 Exercises

[3.4.15] Problem

(1) Let $\mathcal{T}, \mathcal{T}'$ be topologies on X such that $\mathcal{T} \subseteq \mathcal{T}'$. What does compactness of X in one topology say about the other?

(2) Show that if X is compact and Hausdorff under topologies $\mathcal{T}, \mathcal{T}'$, the topologies are either equal or incomparable.

Proof.

(1) If X is compact in $\mathcal{T}'$, then all open covers of X have finite subcovers—necessarily, this includes covers formed by sets in $\mathcal{T}$, and so X is compact in $\mathcal{T}$. No statement can be made in the reverse direction.

(2) Suppose the two topologies are comparable: without loss of generality, say $\mathcal{T} \subseteq \mathcal{T}'$. Then define the map

$$f : (X, \mathcal{T}') \to (X, \mathcal{T}) \quad f(x) = x.$$

Note f is a bijective, continuous map. Notably, it sends closed sets to closed sets: if $C \subseteq X$ is closed, then it is compact, and since f is continuous, $f(C)$ is also compact (and thus closed since the codomain is Hausdorff). Thus f is a homeomorphism and so $\mathcal{T} = \mathcal{T}'$. Thus the topologies are either equal or incomparable.

[3.4.16] Problem

Show that every compact subspace of a metric space is closed and bounded (with respect to the metric). Give an example of a closed and bounded subset of a metric space that is not necessarily compact.

Proof. Suppose (X, d) is a metric space in its metric topology, and suppose $C \subseteq X$ is compact. Note that the metric topology is Hausdorff, and since C is a compact subset of a Hausdorff space, it is closed. To show it is bounded, let $\mathcal{A}$ be an open cover of C with basic, open subsets from X. Then we may take a finite subcover $\{A_1, \dots, A_n\}$. Finitely many open balls can be contained in exactly one ball of finite radius, which means C is bounded.

Consider the metric space $\mathbb{Q}$ together with metric $d(x, y) = |x - y|$. Then note that $[0, \sqrt{2}) \subseteq \mathbb{Q}$ is closed and bounded. Then consider the open cover $\mathcal{A} = (-1, 1) \cup \{(0, \sqrt{2} - 1/n)\}_{n \in \mathbb{Z}_+}$, which has no finite subcover.

[3.4.17] Problem

Suppose A, B are disjoint, compact subspaces of the Hausdorff space X. Show that there exist disjoint open sets U, V such that $A \subseteq U$ and $B \subseteq V$.

Proof. Note that in a Hausdorff space, a compact space and a point outside of it can be separated by two disjoint open neighborhoods. Namely, for each $x \in A$, there are disjoint open neighborhoods $U_x \subseteq X$ of x and $V_x \subseteq X$ of B. Then note $\{U_x\}_{x \in A}$ forms an open cover of A, and by compactness of A, we have a finite subcover $\{U_k\}_{k=1}^n$. Then simply define

$$U = \bigcup_{k=1}^n U_k \quad V = \bigcap_{k=1}^n V_k,$$

which are both open open sets containing A and B respectively. For each $u \in U$, there is some $j \in \{1, \dots, n\}$ such that $u \in U_j$, meaning $u \notin V_j$ and so $u \notin V$. Thus U, V are disjoint, completing the proof.

3.5 Compact Subspaces of the Real Line

[3.5.1] THEOREM (*Closed Intervals are Compact in Order Topology*)

Let X be a totally ordered set with the least upper bound property. In its order topology, closed intervals $[a,b] \subseteq X$ are compact subsets of X.

Proof. We split this proof into four steps. For setup, let $a < b \in X$, and consider the interval $[a,b]$ with an arbitrary open cover $\mathcal{A}$ of $[a,b]$, where the open sets are taken from the subspace topology on $[a,b]$ (which coincides with the order topology). We show there is a finite subcover of $\mathcal{A}$.

(1) First we show that, for any $x \in [a,b)$, there is some $y > x$ such that $[x,y]$ can be covered by, at most, two elements of $\mathcal{A}$. If y is the immediate successor to x, then note that $[x,y] = \{x,y\}$, so there is certainly one element of $\mathcal{A}$ that contains x and one element (not necessarily distinct) of $\mathcal{A}$ that contains y. Now suppose y is not the immediate successor to x. There is some $A \in \mathcal{A}$ such that $x \in A$. Since A is open, there is some $c \in [a,b]$ such that $x \in [x,c) \subseteq A$. Then choose $y \in (x,c)$, and so $[x,y] \subseteq A$.

(2) Let C be the set of points $y > a$ in $[a,b]$ such that $[a,y]$ can be covered by finitely many elements of $\mathcal{A}$. Then the set C is nonempty by (1), using $x = a$ (there is definitely some y such that $[a,y]$ can be covered by up to two elements of $\mathcal{A}$).

(3) Let $c = \sup(C)$—we seek to show $c \in C$. Let $A \in \mathcal{A}$ be some open set containing c; then there is some $d \in [a,b]$ such that $c \in (d,c] \subseteq A$. For the sake of contradiction, suppose $c \notin C$. Then there is some point z of C such that $z \in (d,c)$, since otherwise d would be a smaller upper bound for C. Since $z \in C$, there are finitely many sets from $\mathcal{A}$ that covers $[a,z]$, say n. Then $[z,c] \subseteq A$, so $[a,c] = [a,z] \cup [z,c]$ can be covered by, at most, $n+1$ sets, which is finite. Thus $c \in C$.

(4) Finally, we show $c = b$. Suppose, for the sake of contradiction, that $c < b$. Then, by (1), there is some $y \in (c,b]$ such that $[c,y]$ is covered by up to two sets from $\mathcal{A}$. Then $[a,y] = [a,c] \cup [c,y]$ can be covered by finitely many elements, so $y \in C$, contradicting that c is an upper bound for C.

Thus $[a,b]$ can be covered by finitely many elements from $\mathcal{A}$, completing the proof. ❦

[3.5.2] REMARK (*Intuition Behind Proof*)

This proof is quite long, so we break it down intuitively. Consider any closed interval $[a,b]$ from the space X in its order topology with the least upper bound property. Let $\mathcal{A}$ be an open cover for $[a,b]$ comprising subsets of $[a,b]$ that are open in its subspace topology (which coincides with its order topology). First we establish that, for any point $x \in [a,b)$, there is some element of this interval next to x such that the area between them ($[x,y]$) can be covered by finitely many members of $\mathcal{A}$. Obviously we exclude b, since nothing comes after b. Knowing this, we know that there is some element after a such that the area between them can be covered by finitely many members of $\mathcal{A}$. We let C be all the elements of $[a,b]$ that satisfy this, and since it is nonempty, the value $c = \sup(C)$ is well-defined. For any member of $\mathbb{C}$ that contains c, since it is open, there is some nontrivial intersection with C, say at $z \in C$. By definition of C, we know $[a,z]$ can be finitely covered, and since $[z,c] \subseteq A$, the entire interval $[a,c] = [a,z] \cup [z,c]$ can be finitely covered, and so $c \in C$. Finally, to show $c = b$, we note that if $c < b$, then there is some $y \in (c,b]$ such that $[c,y]$ is finitely covered, and so $[a,y] = [a,c] \cup [c,y]$ is finitely covered, contradicting that c is an upper bound for C. Thus $c = b$, completing the proof.

[3.5.3] COROLLARY

Every closed interval in $\mathbb{R}$ is compact.

[3.5.4] Theorem (*Compact iff. Closed and Bounded*)

A subspace $A \subseteq \mathbb{R}^n$ is compact if and only if it is closed and bounded under either the square or Euclidean metric.

Proof. Note that $\rho(x,y) \le d(x,y) \le \sqrt{n}\rho(x,y)$ for $x, y \in \mathbb{R}^n$, so we only consider the square metric, since A is bounded under the Euclidean metric if and only if it is bounded under the square metric.

($\Longrightarrow$): Suppose A is compact. Since it is a compact subspace of a Hausdorff space, it is automatically closed, so we simply show A is bounded. Let $\{B_\rho(\mathbf{0}_n, m)\}_{m \in \mathbb{Z}_+}$ be an open cover for $\mathbb{R}^n$—then a finite subcover of this covers A. Then a box in $\mathbb{R}^n$ can contain the entirety of A, implying it is bounded.

($\Longleftarrow$): Suppose A is closed and bounded under ρ. Let $N \in \mathbb{R}$ be such that $\rho(x,y) \le N$ for every $x, y \in A$. Choose some $x_0 \in A$, then let $\rho(x_0, \mathbf{0}_n) = b$. Then, for any $x \in A$, note that

$$\rho(x, \mathbf{0}) \le \rho(x, x_0) + \rho(x_0, 0) = N + b.$$

Then A is a closed subset of the compact subspace $B_\rho(\mathbf{0}, N+b)$, implying A is compact. ❦

[3.5.5] Theorem (*Extreme Value Theorem*)

Suppose $f : X \to Y$ is a continuous map from a compact set X to an ordered set Y in its order topology. Then there are $c, d \in X$ such that $f(c) \le f(x) \le f(d)$ for every $x \in X$.

Proof. Consider $A = \operatorname{im}(f) = f(X)$, a compact set by continuity. We show that A contains a maximum M and minimum m, which completes the proof when taking $c \in f^{-1}(\{m\})$ and $d \in f^{-1}(\{M\})$. We show that A contains a maximum and omit the proof for a minimum, as it is entirely symmetric.

For the sake of contradiction, suppose A has no maximum. Then note $\mathcal{A} = \{(-\infty, a)\}_{a \in A}$ is a collection of open subsets of X. For any $y \in A$, note there is some $a \in A$ such that $y < a$, and so $y \in (-\infty, a)$. Thus $\mathcal{A}$ forms an open cover for $f(X)$.Take a finite subcover $\{(-\infty, a_k)\}_{k=1}^n$ with each $a_k \in A$. Choosing $a = \max\{a_k\}_{k=1}^n$, note that $a \in A$ but a is not covered by this finite subcover, a contradiction. Thus A has a maximum. ❦

For $X, Y = \mathbb{R}$, note this is the Extreme Value Theorem from calculus.

We now prove a theorem about uniform continuity: that a continuous map from a compact metric space to a target metric space is uniformly continuous. In the process, we will use something known as a Lebesgue number and the Lebesgue covering lemma.

[3.5.6] Definition (*Distance from Point to Metric Space*)

Suppose (X,d) is a metric space, and let $A \subseteq X$ is nonempty. Then, for any $x \in X$, we define

$$d(x,A) = \inf_{a \in A}\{d(x,a)\}.$$

Generally, $d(x,A)$ is the "shortest" distance from a point to any member of a set A. Contrast this with the *diameter* of A, which is defined as the "largest" distance between any two points in A (that is, the diameter is $\sup_{x,y \in A}\{d(x,y)\}$).

[3.5.7] Theorem

For a metric space (X,d) and a fixed, nonempty subset $A \subseteq X$, the function $d(-,A)$ is continuous.

Proof. Fix $x,y \in X$. Then note

$$d(x,A) \le d(x,a) \le d(x,y) + d(y,a)$$

for each $a \in A$. It follows that

$$d(x,A) - d(x,y) \le \inf_{a \in A}\{d(y,a)\} = d(y,A) \implies d(x,A) - d(y,A) \le d(x,y).$$

Switching the roles of x, y, we arrive at the inequality

$$|d(x,A) - d(y,A)| \le d(x,y).$$

For any $\varepsilon > 0$, choosing $\delta = \varepsilon$ proves $d(-,A)$ is continuous by the $\varepsilon - \delta$ characterization of continuity. ❦

[3.5.8] Theorem (*Lebesgue Covering Lemma*)

Let (X,d) be a compact metric space together with any arbitrary open cover $\mathcal{A}$. Then there exists some $\delta > 0$, called a **Lebesgue number**, such that every nonempty subset of X with diameter less than δ is contained in some member of the cover.

Proof. Let (X,d) be a compact metric space, and let $\mathcal{A}$ be an open cover of X. Then take a finite subcover $\{A_k\}_{k=1}^{n}$. For each $k \in \{1,\dots,n\}$, define $C_k = X \setminus A_k$. Then define a function

$$f : X \to \mathbb{R} \quad f(x) = \frac{1}{n}\sum_{k=1}^{n} d(x, C_k).$$

We show $f(x) > 0$ for every $x \in X$. For fixed $x \in X$, there is some k such that $x \in A_k$. Since A_k is open, we may choose some $\varepsilon > 0$ such that $x \in B(x,\varepsilon) \subseteq A_k$. Then $d(x,C_k) \ge \varepsilon$, and so $f(x) \ge \varepsilon/n$.

Note f is a constant multiple of a sum of continuous functions, and so f is continuous. Thus f attains a minimum value δ, which is greater than 0 by the previous statement, by the Extreme Value Theorem, which we will show is our desired Lebesgue number.

Let $B \subseteq X$ be a nonempty subset whose diameter is less than δ. Choose some $x_0 \in B$, then note B lies in the δ-neighborhood of x_0. Thus

$$\delta \le f(x_0) \le d(x_0, C_m),$$

where C_m is chosen such that it is the maximum of $d(x_0, C_k)$ for each k. Then the δ-neighborhood of x_0 is contained in the element $A_m = X \setminus C_m$ of the covering $\mathcal{A}$, meaning B is as well. ❦

Recall the definition of uniform continuity.

[3.5.9] Definition (*Uniform Continuity*)

Suppose (X, d_X) and (Y, d_Y) are metric spaces. A map $f : X \to Y$ is said to be **uniformly continuous** if, for every $\varepsilon > 0$, there exists some $\delta > 0$ such that for every $x_1, x_2 \in X$, it follows that

$$d_X(x_1, x_2) < \delta \implies d_Y(f(x_1), f(x_2)) < \varepsilon.$$

[3.5.10] Theorem (*Uniform Continuity Theorem*)

Suppose $f : X \to Y$ is a continuous map from the compact metric space (X, d_X) to a target metric space (Y, d_Y). Then f is uniformly continuous.

Proof. Fix $\varepsilon > 0$. Let $\mathcal{A} = \{B_{d_Y}(y, \varepsilon/2)\}_{y \in Y}$ be an open cover for Y, and consider the inverse image $f^{-1}(\mathcal{A})$, which forms an open cover for X. Since X forms a compact metric space, there is a Lebesgue number $\delta > 0$. For any set $\{x_1, x_2\}$ whose diameter is less than δ, there is some $A \in \mathcal{A}$ such that $\{x_1, x_2\} \subseteq A$. Note $A = f^{-1}(B_{d_Y}(y, \varepsilon/2))$ for some $y \in Y$, and so $f(x_1), f(x_2) \in B_{d_Y}(y, \varepsilon/2)$. Thus $d_Y(f(x_1), f(x_2)) < \varepsilon$ for any pair $x_1, x_2 \in X$, completing the proof. ❦

We now prove that the reals are uncountable. Namely, we use the order properties of $\mathbb{R}$, avoiding decimals and binary expansions and the like.

[3.5.11] Definition (*Isolated Point*)

Let X be a topological space. A point $x \in X$ is said to be **isolated** if $\{x\}$ is open. Equivalently, x is isolated if it is not a limit point of X, or if there exists an open neighborhood $O \subseteq X$ of x that has no other points.

[3.5.12] Theorem

Suppose X is a nonempty, compact, Hausdorff space. If X has no isolated points, then it is uncountable.

Proof. First, we claim that for any nonempty open $U \subseteq X$ and $x \in X$, there is a nonempty open $V \subseteq U$ such that $x \notin \overline{V}$. Choose $y \in U$ distinct from x. (If $x \in U$, y exists because x is not isolated; if $x \notin U$, y exists because U is nonempty.) Since X is Hausdorff, there are disjoint open neighborhoods W_1 and W_2 of x and y. Let $V = W_2 \cap U$. Then $V \subseteq U$ is nonempty and open (since $y \in V$). Since W_1 is an open neighborhood of x disjoint from V, $x \notin \overline{V}$.

To show X is uncountable, let $f : \mathbb{Z}_+ \to X$ be any map with $f(n) = x_n$. We show f is not surjective. For $n = 1$, applying the claim to $U = X$ allows us to choose a nonempty open $V_1 \subseteq X$ such that $x_1 \notin \overline{V}_1$. Inductively, given V_{n-1}, applying the claim to $U = V_{n-1}$ allows us to choose a nonempty open $V_n \subseteq V_{n-1}$ such that $x_n \notin \overline{V}_n$.

Thus, $\overline{V}_1 \supseteq \overline{V}_2 \supseteq \cdots$ is a sequence of nonempty closed sets with the finite intersection property. By compactness, $\bigcap_{n \in \mathbb{Z}_+} \overline{V}_n$ is nonempty, so it contains some element x. Since $x_n \notin \overline{V}_n$ for all n, $x \neq x_n$. Thus $x \notin f(\mathbb{Z}_+)$, meaning f is not surjective and X is uncountable. ❦

[3.5.13] Corollary

Every nonempty compact subset of $\mathbb{R}$ is uncountable.

[3.5.14] Recap

(1) Let X be a totally ordered set with the least upper bound property. In its order topology, closed intervals of X are compact subspaces.

> **Intuition.** Let $[a,b] \subseteq X$ be a closed interval, and let $\mathcal{A}$ be an open cover for this interval, where it takes open subsets of $[a,b]$ in its subspace topology. First, we show that for every $x \in [a,b)$, there is some $y > x$ such that $[x,y] \subseteq [a,b]$ can be covered by finitely many members of the cover $\mathcal{A}$ (in fact, by up to two!).
>
> Then we consider starting at a. Let C be the set of all points $y > a$ such that $[a,y] \subseteq [a,b]$ can be covered by finitely many members of $\mathcal{A}$. We are guaranteed that $C \neq \emptyset$, since letting $x = a$ and applying the previous logic proves there is some element in C. Since it is non-empty and bounded, there is some $c = \sup(C)$. Let $A \in \mathcal{A}$ be a member of the cover such that $c \in A$. By A being open, it follows there is some $d < c$ such that $c \in (d,c] \subseteq A$. Note that there is some $z \in C$ such that $d < z < c$, otherwise d would be a smaller upper bound than C. Then $[a,c] = [a,z] \cup [z,c]$, where the first part is covered by finitely many members of $\mathcal{A}$ as per the definition of C, and the second part is covered by finitely many members of $\mathcal{A}$ since $[z,c] \subseteq A$—thus $c \in C$.
>
> Now we show $c = b$. For the sake of contradiction, suppose $c < b$. Following the logic of the previous paragraph, there is some $A \in \mathcal{A}$ such that $c \in A$, and since A is open, there is some $x \in (c,b]$ such that $[c,x] \subseteq A$. Then $[a,x] = [a,c] \cup [c,x]$ can be finitely covered, meaning $x \in C$, contradicting that c is an upper bound. Thus $c = b$.

(2) As a consequence, taking $X = \mathbb{R}$ means are closed intervals in $\mathbb{R}$ are compact.

(3) **Compact iff. Closed and Bounded in $\mathbb{R}^n$.** A subspace $A \subseteq \mathbb{R}^n$ is compact if and only if it is closed and bounded under the Euclidean or square metrics.

> **Intuition.** Note that a subspace $A \subseteq \mathbb{R}^n$ is bounded under the Euclidean metric if and only if it is bounded under the square metric, since
> $$\rho(x,y) \leq d(x,y) \leq \sqrt{n}\rho(x,y)$$
> for every $x, y \in \mathbb{R}^n$.
>
> ($\Longrightarrow$): Suppose A is compact. Then A is a compact subspace of the Hausdorff space $\mathbb{R}^n$, and so A is closed. Let $\{B_\rho(\mathbf{0}_n, m)\}_{m \in \mathbb{Z}_+}$ be an open cover for $\mathbb{R}^n$, then choose a finite subcover that covers A. Then the union of all members of the finite subcover is contained in a box, and since A is inside this box, A is bounded as well.
>
> ($\Longleftarrow$): Suppose A is closed and bounded under ρ. Suppose that $N \in \mathbb{R}$ is such that $\rho(x,y) \leq N$ for each $x, y \in A$. Fix some $x_0 \in A$, then let $\rho(x_0) = b$. Then note $\rho(x, \mathbf{0}_n) \leq \rho(x, x_0) + \rho(x_0, \mathbf{0}_n) = N + b$ for every $x \in A$, and so $A \subseteq B_\rho(\mathbf{0}_n, N+b)$. Since A is a closed subset of the compact box, A is compact as well.

(4) **Extreme Value Theorem.** Let $f : X \to Y$ be a continuous map from a topological space X to a totally ordered set Y in its order topology. If X is compact, there exist $c, d \in X$ such that $f(c) \leq f(x) \leq f(d)$ for every $x \in X$.

> **Intuition.** This theorem essentially states that $f(X)$ has a minimum and maximum. We prove $f(X)$ contains its maximum, since the proof that it contains its minimum is analogous. For the sake of contradiction, we suppose $f(X)$ does not contain a maximum. Then note $\{(-\infty, a)\}_{a \in f(X)}$ is a set of open subsets of Y. Since $f(X)$ has no maximum, for each $y \in f(X)$, there is some $a > y \in f(X)$, so this collection forms an open cover for $f(X)$. Since X is compact, $f(X)$ is also compact by continuity of f. Thus there is a finite subcover
> $$\{(-\infty, a_1), \ldots, (-\infty, a_n)\}$$
> that covers $f(X)$. Note $a = \max\{a_1, \ldots, a_n\}$ exists and should be an element of A, but it is not contained in the finite subcover, a contradiction.

(5) For a metric space (X,d) and some nonempty subset $A \subseteq X$, we define the distance of a point $x \in X$ from A by $d(x,A) = \inf_{a \in A}\{d(x,a)\}$. Colloquially, this is the shortest distance from the point to the set A. Compare this to the diameter of A, which is colloquially the largest distance between any two points in A.

(6) The function $d(-,A)$ is continuous.

> **Intuition.** Let $x, y \in X$ be arbitrary points. Note then that
> $$d(x,A) \le d(x,a) \le d(x,y) + d(y,a)$$
> for every $a \in A$. Notably, we seek to minimize $d(y,a)$, such that
> $$d(x,A) \le d(x,y) - d(y,a) \le d(x,y) - \inf_{a \in A}\{d(y,a)\} \implies d(x,A) - d(y,A) \le d(x,y).$$
> Interchanging the role of x, y yields $|d(x,A) - d(y,A)| \le d(x,y)$, implying $d(-,A)$ is Lipschitz and thus continuous.

(7) **Lebesgue Covering Lemma.** Suppose (X,d) is a compact metric space together with an open cover $\mathcal{A}$. Then there exists some $\delta > 0$, called a **Lebesgue number**, such that every set whose diameter is less than δ is contained in one member of the cover.

> **Intuition.** Since X is compact, it admits a finite subcover $\{A_k\}_{k=1}^n$, where each $A_k \in \mathcal{A}$. Let $C_k = X \setminus A_k$ for each k, then define the function $f : X \to \mathbb{R}$ by
> $$f(x) = \frac{1}{n}\sum_{k=1}^{n} d(x, C_k).$$
> We demonstrate that $f(x) > 0$ for every $x \in X$. For a fixed $x \in X$, there is some k such that $x \in A_k$. Since A_k is open, there is some $\varepsilon > 0$ such that $x \in B(x,\varepsilon) \subseteq A_k$. Thus $d(x, C_k) \ge \varepsilon$ and so $f(x) \ge \varepsilon/n$. Since X is compact, f attains a minimum value $\delta > 0$ (where $\delta > 0$ since $f(x) > 0$) by EVT. We show this is our desired Lebesgue number.
>
> Consider any set B whose diameter is δ. Then there is some $x_0 \in B$ such that $B \subseteq B(x_0, \delta)$. Then note
> $$\delta \le f(x_0) \le d(x, C_m),$$
> where m is chosen to maximize the value of $d(x, C_k)$ for each k. Then $B(x_0,\delta) \subseteq A_m$, where $A_m = X \setminus C_m$, implying $B \subseteq A_m \in \mathcal{A}$, completing the proof.

(8) **Uniform Continuity Theorem.** Suppose $f : X \to Y$ is a continuous map from a compact metric space (X, d_X) to a target metric space (Y, d_Y). Then f is uniformly continuous, meaning that for any $\varepsilon > 0$, there is some $\delta > 0$ such that
$$d_X(x_1,x_2) < \delta \implies d_Y(f(x_1), f(x_2)) < \varepsilon$$
for every $x_1, x_2 \in X$.

> **Intuition.** Fix $\varepsilon > 0$. Let $\mathcal{A} = \{B_{d_Y}(y, \varepsilon/2)\}_{y \in Y}$ be an open cover of $f(X)$ from open subsets of Y. Then note $f^{-1}(\mathcal{A})$ forms an open cover of X. Since X forms a compact metric space, there is a Lebesgue number δ. Let $x_1, x_2 \in X$ be such that $d_X(x_1,x_2) < \delta$. Then $\{x_1, x_2\}$ is a set whose diameter is less than δ, meaning $\{x_1,x_2\} \subseteq A$ for some $A \in \mathcal{A}$. This implies $\{x_1, x_2\} \subseteq f^{-1}(B_{d_Y}(y, \varepsilon/2))$ for some $y \in Y$, and so $\{f(x_1), f(x_2)\} \subseteq B_{d_Y}(y, \varepsilon/2)$. Thus $d_Y(f(x_1), f(x_2)) < \varepsilon$, completing the proof.

(9) For any topological space X, a point $x \in X$ is said to be **isolated** if $\{x\}$ is open. Equivalently, x is isolated if an open neighborhood of x exists that comprises only x, or if x is not a limit point of X.

(10) Suppose X is a nonempty, compact, Hausdorff space. If X has no isolated points, then X is uncountable.

Intuition. The intuition behind this proof is to construct a sequence of nested, open sets $V_1 \supseteq V_2 \supseteq \cdots$ with the finite intersection property, such that $x_n \in V_n$ but $x_n \notin \overline{V}_{n+1}$ for every $n \in \mathbb{Z}_+$, where $x_n \in X$. Then, taking the closure of each V_k (note that if $V_k \subseteq V_{k+1}$, then $\overline{V}_k \subseteq \overline{V}_{k+1}$), we yield a sequence of nested, closed sets with the finite intersection property. Thus $x \in \bigcup_{k \in \mathbb{Z}_+} \overline{V}_k$, but $x \neq x_n$ for each n, completing the proof that X is uncountable.

Let X be a nonempty, compact, Hausdorff space with no isolated points. Let $U \subseteq X$ be a nonempty, open set and let $x \in X$ be arbitrary. We seek to show there is a nonempty, open subset $V \subseteq U$ such that $x \notin \overline{V}$. Choose a point $y \in U$ such that $x \neq y$—if $x \in U$, then U necessarily contains a distinct point y to prevent $\{x\}$ from being open (since X has no isolated points); otherwise if $x \notin U$, then y is guaranteed since U is nonempty. By Hausdorffness of X, there exists open neighborhoods $W_1, W_2 \subseteq X$ of x and y repsectively. Then $V = W_2 \cap U$ is a nonempty, open subset of U. Note $x \notin \overline{V}$ since W_1 is an open neighborhood of X that doesn't intersect W_2, so x is not a limit point of V.

Consider the subset $\{x_n\}_{n \in \mathbb{Z}_+} \subseteq X$. Let $V_1 = X$, and let $V_{k+1} \subseteq V_k$ be nonempty and open such that $x \notin V_{k+1}$ (existence is proved by the last paragraph). Then the closure of each V_k forms a nested sequence of closed sets with the finite intersection property, so there is some $x \in \cup_{k \in \mathbb{Z}_+} \overline{V}_k$. Note that, for each $n \in \mathbb{Z}_+$, $x_n \notin \overline{V}_n$ and so $x \neq x_n$. Thus x is an element of X outside this subset, making X uncountable.

(11) As a direct corollary, it follows that $\mathbb{R}$ is uncountable.

3.5.1 Exercises

[3.5.15] Problem

Suppose X is a totally ordered set in its order topology. Show that if every closed interval of X is compact, then X has the least upper bound property.

Proof. Let $A \subseteq X$ be an arbitrary nonempty, bounded set. Let $a \in A$ be arbitrary, and let b be an arbitrary upper bound. Then consider the interval $[a, b] \subseteq X$. For the sake of contradiction, suppose A has no supremum. Then define an open cover for X by assigning a set to each point $x \in [a, b]$ and forming a collection of sets from it:

(1) If x is not an upper bound: choose some $a' \in A$ such that $x < a'$, then choose the open set $(-\infty, a')$.

(2) If x is an upper bound: choose some upper bound b' such that $b' < x$, then choose the open set (b', ∞).

Note these steps are justified since A has no supremum (i.e., $x \in [a, b]$ cannot be the largest element of A or the least upper bound). Then there is a finite subcover whose union contains $[a, b]$: that is,

$$[a, b] \subseteq \left[\bigcup_{k=1}^{m} (-\infty, a_k)\right] \cup \left[\bigcup_{k=1}^{n} (b_k, \infty)\right]$$

for some $a_k \in A$ for each $k \in \{1, \ldots, m\}$ and for some upper bound b_k of A for each $k \in \{1, \ldots, n\}$. We have that $b_{\min} = \min\{b_k\}_{k=1}^{n}$ is the smallest upper bound of A contained in $[a, b]$, and by convexity of $[a, b]$, we note $b_{\min}$ is the supremum for A, a contradiction. ❦

[3.5.16] Problem

Let (X,d) be a metric space, and let $A \subseteq X$ be nonempty.

(1) Show $d(x,A) = 0$ if and only if $x \in \overline{A}$.

(2) Suppose A is compact. Show that $d(x,A) = d(x,a)$ for some $a \in A$.

(3) Define the ε-neighborhood of A in X to be the set

$$U(A,\varepsilon) = \{x \in X : d(x,A) < \varepsilon\}.$$

Show that $U(A,\varepsilon) = \bigcup_{a\in A} B(a,\varepsilon)$.

Proof.

(1) ($\Longrightarrow$): Suppose $d(x,A) = 0$, meaning $\inf_{a\in A}\{d(x,a)\} = 0$. Let $\varepsilon > 0$ be arbitrary, then consider the open neighborhood $B(x,\varepsilon)$ of x. Since $d(x,A) = 0$, there is some $a \in A$ such that $d(x,a) < \varepsilon$. Thus $a \in B(x,\varepsilon)$, meaning $B(x,\varepsilon) \cap A \neq \emptyset$. Since every arbitrary basic open neighborhood of x intersects A, it follows that $x \in \overline{A}$.

($\Longleftarrow$): Suppose $x \in \overline{A}$. Fix some $n \in \mathbb{Z}_+$, then let $\varepsilon = 1/n$. Note then that $B(x,\varepsilon) \cap A \neq \emptyset$ by definition of $\overline{A}$, so there is some $a_n \in B(x,\varepsilon) \cap A$. Thus $d(x,a_n) < \varepsilon$, and as we let $n \to \infty$, we note $\varepsilon \to 0$. Thus $d(x,A) = 0$.

(2) Define $f : A \to \mathbb{R}$ by $f(a) = d(x,a)$, with $x \in X$ fixed. Let $a_1, a_2 \in A$ be arbitrary, then note

$$d(x,a_2) \leq d(x,a_1) + d(a_1,a_2) \implies d(x,a_2) - d(x,a_1) \leq d(a_1,a_2).$$

Reversing the roles of a_1, a_2, we arrive at $|d(x,a_2) - d(x,a_1)| \leq d(a_1,a_2)$, implying $d(x,-)$ is Lipschitz and thus continuous. Since f is continuous and A is compact, we have that $f(A) = \{d(x,a) : a \in A\}$ takes on a minimum value by the Extreme Value Theorem. Thus there is some $a \in A$ such that $f(a) = d(x,a) = \inf_{a\in A}\{d(x,a)\} = d(x,A)$.

(3) Suppose $x \in U(A,\varepsilon)$, and so $d(x,A) < \varepsilon$. Then there is some $a \in A$ such that $d(x,a) < \varepsilon$, and so $x \in B_d(a,\varepsilon) \subseteq \bigcup_{a\in A} B_d(a,\varepsilon)$. Conversely, suppose $x \in \bigcup_{a\in A} B_d(a,\varepsilon)$. Then, for some $a \in A$, we have that $d(x,A) \leq d(x,a) < \varepsilon$, so $x \in U(A,\varepsilon)$.

[3.5.17] Problem

Suppose (X,d) is a connected metric space. Show that if X has more than one point, then X is uncountable.

Proof. For fixed $x \in X$, define $f : X \to \mathbb{R}$ by $f(y) = d(x,y)$. We have that f is continuous, and since X is connected, the set $f(X)$ is connected as well. Note connected subsets of $\mathbb{R}$ coincide with intervals or rays, which are uncountable if they have at least one element, and so $f(X)$ is uncountable (since X has at least one element). Note that $|X| \geq |f(X)|$, so X is uncountable as well.

3.6 Limit Point Compactness

[3.6.1] Definition (*Limit Point Compactness*)

A space X is said to be **limit point compact** if every infinite subset of X has a limit point in X.

[3.6.2] Theorem (*Compactness $\implies$ Limit Point Compactness*)

A compact space is limit point compact.

> **Proof.** Suppose X is compact. We seek to show if $A \subseteq X$ is infinite, then it contains a limit point in X. We prove by contrapositive: that if A has no limit points, then it is finite. Indeed, take A to have no limit points. Then A is trivially closed, since it contains all its limit points. Since A is a closed subset of a compact space, we have that A is compact as well. For each $a \in A$, note that $U_a = \{a\}$ forms an open neighborhood of a in A, and so $\{U_a\}_{a \in A}$ is an open cover for A. Since A is compact, we have that $\{U_k\}_{k=1}^{n}$ forms a finite subcover, meaning A has finitely many elements. ❦

[3.6.3] Example (*Limit Point Compact Set That Isn't Compact*)

Note the converse is not true in general. Consider the set $Y = \{a, b\}$ with the indiscrete topology. Then consider the set

$$X = \mathbb{Z}_+ \times Y = \bigcup_{n \in \mathbb{Z}_+} \{(n, a), (n, b)\}.$$

Let $U \subseteq X$ be nonempty. If U has any any integer paired with either a or b, then the same integer paired with the other element is a limit point. Since all nonempty subsets of X have a limit point, we have that X is limit point compact. However, note the open covering $\{\{n\} \times Y\}_{n \in \mathbb{Z}_+}$ for X has no finite subcover, and so X is not compact.

[3.6.4] Definition (*Sequential Compactness*)

A topological space X is said to be **sequentially compact** if, for every sequence $(x_n) \subseteq X$, there is some subsequence (x_{n_k}) that converges in X.

In the context of metric spaces, we note that compactness, limit point compactness, and sequential compactness all coincide.

[3.6.5] THEOREM (*Compactness* $\iff$ *Limit Point Compactness* $\iff$ *Sequential Compactness in Metric Spaces*)

Suppose X is a metrizable topological space. Then the following are equivalent.

(1) X is compact.

(2) X is limit point compact.

(3) X is sequentially compact.

Proof. (1) $\implies$ (2): We have proven this already.

(2) $\implies$ (3): Suppose X is limit point compact. Let x_n be an arbitrary sequence in X, and let $A = \{x_n\}_{n\in\mathbb{Z}_+}$. If A is finite, then there is one number that repeats a countably infinite number of times in the sequence, so choose this as a subsequence and note it converges. Otherwise, if A is infinite, then by hypothesis we have a limit point x. For each $k \in \mathbb{Z}_+$, let n_k both be greater than all n_j for $j < k$ and be such that $x_{n_k} \in B(x, 1/k)$. Then $(x_{n_k}) \to x$, and so X is sequentially compact.

(3) $\implies$ (1): Suppose X is sequentially compact. We prove that the Lebesgue covering lemma applies to X, then we prove that finitely many ε-balls can cover X, then we finally prove X is compact.

Let $\mathcal{A}$ be an arbitrary open cover of X. For the sake of contradiction, suppose the Lebesgue covering lemma doesn't apply to X. Then, for each $n \in \mathbb{Z}_+$, there exists a nonempty set C_n whose diameter is less than $1/n$ and does not lie in a single member of $\mathcal{A}$. Choosing $x_n \in C_n$ for each n and forming a sequence, we use our hypothesis to get a convergent subsequence $(x_{n_k}) \to a$. Note that a belongs to some member A of the open cover $\mathcal{A}$, and since A is open, there is some $\varepsilon > 0$ such that $B(a, \varepsilon) \subseteq A$. We may choose an index i large enough such that $1/n_i < \varepsilon/2$, and since $\operatorname{diam}(C_{n_i}) < \varepsilon/2$, we have that $C_{n_i} \subseteq B(x, \varepsilon/2)$; moreover, we may choose i large enough such that $d(x_{n_i}, a) < \varepsilon/2$. For any point $c \in C_{n_i}$, note that

$$d(c,a) \le d(c,x) + d(x,a) < \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

and so $C_{n_i} \subseteq B(a, \varepsilon)$. Thus $C_{n_i} \subseteq A$, contrary to assumption.

Suppose, for the sake of contradiction, that X cannot be covered by finitely many ε-balls, for a fixed $\varepsilon > 0$. Let $x_1 \in X$, then note $B(x_1, \varepsilon) \neq X$, otherwise X could be covered by finitely (namely one) ε-ball. Thus choose $x_2 \in X \setminus B(x_1, \varepsilon)$, and note $B(x_1, \varepsilon) \cup B(x_2, \varepsilon) \neq X$ for the same reason as before. In general, given distinct points $x_1, x_2, \ldots, x_n$ of X, choose x_{n+1} such that $x_{n+1} \notin \bigcup_{k=1}^{n} B(x_k, \varepsilon)$. By construction, we have that $d(x_{n+1}, x_i) \ge \varepsilon$ for every $i \in \{1, \ldots, n\}$, so there is no chance of (x_n) having a convergent subsequence, contrary to the fact that X is sequentially compact.

Finally, we show X is compact. Let $\mathcal{A}$ be an arbitrary open cover ofr X. Since X is sequentially compact, it has a Lebesgue number δ. Choose $\varepsilon = \delta/3$, then note that we may cover X with finitely many ε-balls. Each of these balls has a diameter of $2\delta/3$, so each ball fits in one member of the open cover. There are finitely members of the cover that contain all the balls, which cover X, and so X is compact. ❦

[3.6.6] Recap

(1) A set X is limit point compact if, for every infinite subset $A \subseteq X$, A has a limit point in X.

(2) If X is compact, then X is limit point compact.

> **Intuition.** Suppose X is compact, and let $A \subseteq X$ be an arbitrary subset. To prove X is limit point compact, we need to show that if A is infinite, then it has a limit point. We proceed by contrapositive, so suppose A has no limit points. Vacuously, A contains all its limit points, so A is closed. Since A is a closed subset of a compact space, A is also compact. Note that A has no limit points, so all its elements are isolated, meaning $U_a = \{a\}$ forms an open cover for A for each $a \in A$. Since A is compact, there is a finite subcover $\{U_k\}_{k=1}^n$ comprised of singletons for A, so A is finite (i.e., not infinite).

(3) Limit point compactness does not imply compactness in general. Consider $Y = \{a, b\}$ with the indiscrete topology and $X = \mathbb{Z}_+ \times Y$. Then X is limit point compact (each $\{n, a\}$ is a limit point of $\{n, b\}$ and vice versa), but it is not compact (consider the open cover $\{\{n\} \times Y\}_{n \in \mathbb{Z}_+}$ for X, and note it has no finite subcover).

(4) A set X is **sequentially compact** if every sequence in X has a convergent subsequence.

(5) In a metrizable space, compactness, limit point compactness, and sequential compactness are all equivalent.

> **Intuition.** Our proof of compactness implying limit point compactness is done.
>
> To show limit point compactness implies sequential compactness, we take some sequence $(x_n) \subseteq X$ and consider $A = \{x_n\}_{n \in \mathbb{Z}_+}$. If A is finite, then one term in the sequence repeats countably infinite amount of times, so a constant subsequence made from it converges, of course. Otherwise, if A is finite, it has a limit point a. For each $k \in \mathbb{Z}_+$, we may define an increasing sequence of indices n_k such that $x_{n_k} \in B(a, 1/k)$, and so $(x_{n_k}) \to a$.
>
> Suppose X is sequentially compact.
>
> (a) We first show X admits the Lebesgue covering lemma by contradiction. If it didn't, then—given a cover $\mathcal{A}$—we could define a sequence of nonempty sets $C_n \subseteq X$ such that $\operatorname{diam}(C_n) < 1/n$ and C_n doesn't lie in a single member of the cover. We may define a sequence (x_n) such that $x_n \in C_n$ for each $n \in \mathbb{Z}_+$, then note there is a convergent subsequence $(x_{n_k}) \to a$. Then a lies in a single member A of the open cover $\mathcal{A}$, and since A is open, there is some $\varepsilon > 0$ such that $a \in B(a, \varepsilon) \subseteq A$. Choose an index i large enough such that $1/n_i < \varepsilon/2$ (and so $\operatorname{diam}(C_{n_i}) < \varepsilon/2$, meaning $C_{n_i} \subseteq B(x, \varepsilon/2)$), and choose i large enough so that $d(x_{n_i}, a) < \varepsilon/2$. Then, for any $c \in C_{n_i}$, we have that
>
> $$d(c, a) \le d(c, x) + d(x, a) < \varepsilon/2 + \varepsilon/2 = \varepsilon,$$
>
> so $C_{n_i} \subseteq B(a, \varepsilon) \subseteq A$, a contradiction.
>
> (b) Next, we show that X can be covered by finitely many ε-balls by contradiction. Fix $\varepsilon > 0$, and suppose the result is false. Choose $x_1 \in X$, then note $B(x_1, \varepsilon) \neq X$ (since finitely many ε-balls may not cover X), so there is some $x_2 \in X \setminus B(x_1, \varepsilon)$. Proceed inductively: that is, for distinct points $x_1, \ldots, x_n \in X$, there is some x_{n+1} such that $x_{n+1} \notin \bigcup_{k=1}^n B(x_n, \varepsilon)$. Then $d(x_{n+1}, x_i) \ge \varepsilon$ for every $i \in \{1, \ldots, n\}$, and so (x_n) has no convergent subsequences, a contradiction.
>
> (c) Finally, we show X is compact. Let $\mathcal{A}$ be an open cover for X, and choose a Lebesgue number $\delta > 0$. Let $\varepsilon = \delta/3$, then note there are finitely many ε-balls that cover X. Each ε-ball has diameter $2\delta/3 < \delta$, and so each belongs to one member of $\mathcal{A}$. Since there are finitely many of them, we choose a finite subcover of $\mathcal{A}$ that encompasses all the finitely many ε-balls that cover X, completing the proof.

3.6.1 Exercises

[3.6.7] Problem

Show that $[0,1]^\omega$ equipped with the uniform topology is not limit point compact.

> **Proof.** We seek to find some infinite subset $A \subseteq [0,1]^\omega$ with no limit points. Consider the set of all standard basis vectors of $\mathbb{R}^\omega$—that is, the set $A = \{e_n\}_{n\in\mathbb{Z}_+}$, where $e_n : \mathbb{Z}_+ \to [0,1]$ is defined by $e_n(j) = \delta_{n,j}$. Then, for any e_j, e_k for $j \neq k \in \mathbb{Z}_+$, we have that
> $$\overline{\rho}(e_j, e_k) = \sup_{n\in\mathbb{Z}_+} |x_n - y_n| = 1.$$
> For the sake of contradiction, suppose $x \in [0,1]^\omega$ is a limit point. Then $B(x,1/3) \cap [A \setminus \{x\}] \neq \emptyset$. In fact, $B(x,1/3)$ intersects A infinitely many times outside of x. If $B(x,1/3) \cap [A \setminus \{x\}] = \{e_1, \ldots, e_n\}$, then we could take $r = \min\{d(x,e_k)\}_{k=1}^n$ and choose $r' < r$ so that $B(x,r') \cap [A \setminus \{x\}] = \emptyset$ (and so x not a limit point), a contradiction. Thus let $e_1, e_2 \in B(x,1/3) \cap [A \setminus \{x\}]$. Then note
> $$\overline{\rho}(e_1, x) \geq \overline{\rho}(e_1, e_2) - \overline{\rho}(x, e_2) = 1 - \underbrace{\overline{\rho}(x, e_2)}_{\leq 1/3} \geq 2/3,$$
> which means $e_1 \notin B(x,1/3)$, a contradiction. ❦

[3.6.8] Problem

Suppose X is limit point compact.

(1) Suppose $f : X \to Y$ is continuous. Is $f(X)$ limit point compact?

(2) If $A \subseteq X$ is closed, is A limit point compact?

(3) If X is a subspace of a Hausdorff space Z, is X closed?

> **Proof.**
>
> (1) No. Let $X = \mathbb{Z}_+ \times Y$, where $Y = \{a,b\}$ with the indiscrete topology. Then define $f : X \to \mathbb{Z}_+$ (where $\mathbb{Z}_+$ takes its subspace topology from $\mathbb{R}$) by $(n,*) \mapsto n$. Note that any singleton subset $\{n\} \subseteq \mathbb{Z}_+$ is a basic, open set, so showing $f^{-1}(\{n\})$ is open guarantees continuity. Indeed, the inverse image is $\{n\} \times Y$, which is an open subset of X. Of course, $\mathbb{Z}_+$ is not limit point compact in its metric topology, completing the disproof.
>
> (2) Yes. Suppose $U \subseteq A$ is infinite. Since $U \subseteq X$, we have that U has a limit point $x \in X$. Since A contains all its limit points, it follows that $x \in A$, and so A is limit point compact.
>
> (3) No. For our counterexample, we want A to have a limit point in $Z \setminus X$, meaning X is not closed (it does not contain all of its limit points). Consider $X = S_\Omega$, which is limit point compact, and $Z = \overline{S}_\Omega$, which is Hausdorff (all order topologies are Hausdorff). Note the subset $X \subseteq X$ has a limit point Ω, but $\Omega \notin X$, and so X is not closed. ❦

[3.6.9] Problem

A space X is said to be **countably compact** if, for any countable family of open sets that cover X, a finite subcover exists. Show that if X is T_1, then X is countably compact if and only if it is limit point compact.

Proof.

($\Longrightarrow$): Suppose X is T_1 and countably compact: we proceed by contrapositive. Suppose $A \subseteq X$ has no limit points: we show that A is finite. Indeed, note that A contains all of its limit points, and so it is closed and contains only isolated points. Thus singleton subsets $\{*\} \subseteq A$ are open in A. For the sake of contradiction, suppose A is infinite. Then we may choose a countably infinite set of distinct points $[A' = \{x_n\}_{n \in \mathbb{Z}_+}] \subseteq A$. Consider the countably infinite cover of X formed by taking each singleton subset of A' and adjoining it to $X \setminus A'$. By countable compactness of X, we take a finite subcover; discarding the $X \setminus A'$ element, we get a covering of A by finitely many singletons, a contradiction. Thus A is finite, meaning X is limit point compact.

($\Longleftarrow$): Suppose X is T_1 and limit point compact: we show X is countably compact by contradiction. Suppose X is not countably compact, and take a countable open cover $\mathcal{U} = \{U_n\}_{n \in \mathbb{Z}_+}$ of X. Note that there is no finite subcover, so for each $n \in \mathbb{Z}_+$, choose $x_n \notin U_1 \cup \cdots \cup U_n$. Since $X = \bigcup U_n$, the set $A = \{x_n\}_{n \in \mathbb{Z}_+}$ is infinite. By limit point compactness, A has a limit point x. Since $\mathcal{U}$ covers X, $x \in U_k$ for some $k \in \mathbb{Z}_+$. Since X is T_1, the neighborhood U_k of the limit point x must contain infinitely many points of A. However, by construction, $x_n \notin U_k$ for all $n \geq k$, implying $U_k \cap A \subseteq \{x_1, \ldots, x_{k-1}\}$. This finite intersection contradicts x being a limit point. Thus, X is countably compact. ❦

[3.6.10] Problem

Suppose (X, d) is a metric space. If $f : X \to X$ is such that

$$d(f(x), f(y)) = d(x, y),$$

we say f is an **isometry**.

(1) Show that f is an embedding.

(2) If X is compact, show that f is a homeomorphism.

Proof.

(1) Suppose $f(x) = f(y)$ for some $x, y \in X$. Then note $0 = d(f(x), f(y)) = d(x, y)$, meaning $x = y$ and so f is injective. Next, choose $\varepsilon > 0$, and consider the ball $O = B(x, \varepsilon) \subseteq X$, a basic open set. For any $y \in O$, note that $d(f(x), f(y)) = d(x, y)$, so $f(y) \in B(f(x), \varepsilon)$. Indeed, $f(O) = B(f(x), \varepsilon) \cap f(X)$, meaning $f^{-1}|_{f(X)}$ is continuous. Showing $f : X \to f(X)$ is continuous follows almost exactly the same idea. Thus f is a homeomorphism onto its image, i.e. is an embedding.

(2) By continuity of f, we have that $f(X)$ is compact. Note that $f(X) \subseteq X$ is a compact subspace of a Hausdorff space, and so $f(X)$ is closed. For the sake of contradiction, suppose $a \in X \setminus f(X)$. Since a belongs to an open set, there is some $\varepsilon > 0$ such that $B(a, \varepsilon) \cap f(X) = \emptyset$. Note then that $d(a, f(a)) \geq \varepsilon$. Moreover, $d(f(a), f(f(a))) = d(a, f(a)) \geq \varepsilon$, where the equality is by hypothesis. Indeed, for $x_1 = a$ and $x_{n+1} = f(x_n)$, we have that $d(x_m, x_n) \geq \varepsilon$ for every $m \neq n$. Note the sequence $\{x_n\}_{n \in \mathbb{Z}_+ \setminus \{1\}} \subseteq f(X)$ has no convergent subsequence, and so $f(X)$ is not sequentially compact, a contradiction (since compactness and sequential compactness are equivalent in metrizable spaces). Thus $X \setminus f(X) = \emptyset$, and so $f(X) = X$, meaning f is a surjective embedding and thus a homeomorphism.

❦

[3.6.11] Problem

Let (X,d) be a metric space. If $f: X \to X$ satisfies the condition

$$d(f(x), f(y)) < d(x,y) \quad \forall x \neq y \in X,$$

then f is a **shrinking map**. If there is a number $C < 1$ such that

$$d(f(x), f(y)) \leq C d(x,y) \quad \forall x \neq y \in X,$$

then f is a **contraction**. A **fixed point** of f is a point x^* such that $f(x^*) = x^*$.

(1) If $f: X \to X$ is a contraction with X compact, show f has a unique fixed point.

(2) If $f: X \to X$ is a shrinking map with X compact, show f has a unique fixed point.

Proof.

(1) Let $A_n = f^n(X)$, where f^n denotes applying the function n times. Note that $f: X \to X$, and so $f(X) \subseteq X$ since $f(X)$ is a subset of the codomain. We will show $f(f(X)) \subseteq f(X)$, and then by induction, it will follow that $A_{n+1} \subseteq A_n$ for any $n \in \mathbb{Z}_+$. If $y \in f(f(X))$, there is some $x \in f^{-1}(\{y\}) \in f(X) \subseteq X$, and so $y = f(x) \in f(X)$ as desired.

Note that contractions are Lipschitz and thus continuous. Since X is compact, each A_n is compact, and as a subspace of a metric (thus Hausdorff) space, each A_n is closed. Since A_n forms a sequence of nested closed sets, by compactness of X we have that there is some $x^* \in \bigcap_{n \in \mathbb{Z}_+} A_n$. We show x^* is our desired fixed point.

Let $A = \bigcap_{n \in \mathbb{Z}_+} A_n$, and note $x^* \in A$. Note that all potential fixed points must live in A, so to show x^* is unique, we show A has only one point. Since $A_{n+1} \subseteq A_n$, we have that $\operatorname{diam}(A_{n+1}) \leq \operatorname{diam}(A_n)$. Since X is a compact metric space, we have that X is bounded, meaning $\operatorname{diam}(X) = M$ for some $M \in \mathbb{R}$. Then $\operatorname{diam}(A_n) = \operatorname{diam}(f^n(X)) \leq C^n M$, and as $n \to \infty$, we have $\operatorname{diam}(A_n) \to 0$, implying $\operatorname{diam}(A) = 0$. Thus $A = \{x^*\}$.

Finally, to show x^* is a fixed point, we show $f(A) \subseteq A$. If $y \in f(A)$, then $y = f(x^*)$. Since $x^* \in A_n$ for each $n \in \mathbb{Z}_+$, we have that $f(x^*) \in f(A_n) = A_{n+1} \subseteq A_n$ for each $n \in \mathbb{Z}_+$. Thus $y \in A$, meaning $f(\{x^*\}) \subseteq \{x^*\}$. This forces $f(x^*) = x^*$, meaning x^* is a fixed point of f, and by the previous argument, this fixed point is unique.

(2) Let $A_n = f^n(X)$ and $A = \bigcap_{n \in \mathbb{Z}_+} A_n$. As in the previous part, f is continuous (since it is a shrinking map, it is 1-Lipschitz), each A_n is closed and nested, and A is a nonempty compact set with $f(A) \subseteq A$. We show $A \subseteq f(A)$.

Let $x \in A$. For each $n \in \mathbb{Z}_+$, since $x \in A_{n+1} = f^{n+1}(X)$, there exists $x_n \in X$ such that $x = f^{n+1}(x_n)$. Define $y_n = f^n(x_n)$, noting $y_n \in A_n$ and $f(y_n) = x$ for all n. Since X is sequentially compact, (y_n) has a convergent subsequence $(y_{n_k}) \to a \in X$. For any fixed $m \in \mathbb{Z}_+$, the tail of the subsequence lies entirely in the closed set A_m, so $a \in A_m$. Thus $a \in A$. By continuity of f, $f(a) = \lim_{k\to\infty} f(y_{n_k}) = \lim_{k\to\infty} x = x$. Thus $x \in f(A)$, so $A = f(A)$.

We now show $\operatorname{diam}(A) = 0$. Since A is compact, if $\operatorname{diam}(A) > 0$, the continuous function d attains a maximum on $A \times A$, so there exist $p, q \in A$ such that $d(p,q) = \operatorname{diam}(A)$. Since $A = f(A)$, we can choose $u, v \in A$ such that $f(u) = p$ and $f(v) = q$. Since $p \neq q$, we must have $u \neq v$. Because f is a shrinking map,

$$\operatorname{diam}(A) = d(p,q) = d(f(u), f(v)) < d(u,v) \leq \operatorname{diam}(A),$$

which is a contradiction. Thus $\operatorname{diam}(A) = 0$, meaning $A = \{x^*\}$.

Since $f(A) = A$, we have $f(x^*) = x^*$. If y is another fixed point, $d(x^*, y) = d(f(x^*), f(y)) < d(x^*, y)$, a contradiction. Thus x^* is unique.

3.7 Local Compactness

[3.7.1] Definition (*Local Compactness*)

A space X is said to be **locally compact at** x if there exists some compact set $C \subseteq X$ that contains an open neighborhood of x. If each point of X is locally compact, X is said to be **locally compact**.

[3.7.2] Example (*Examples of Locally Compact Spaces*)

(1) Any compact space X is locally compact, since for each $x \in X$, there is some open neighborhood O of x such that $x \in O \subseteq X$, with X being the compact superspace.

(2) $\mathbb{R}$ is locally compact. For each $x \in \mathbb{R}$, there is some $a, b \in \mathbb{R}$ such that $x \in (a, b) \subseteq [a, b]$, where $[a, b]$ is compact. The same is true of $\mathbb{R}^n$, where instead we can draw an open ball around a point x that is contained inside a closed one.

(3) Any ordered set X is compact since every basis element is contained in a closed interval.

(4) $\mathbb{R}^\omega$ in its product topology is *not* locally compact, since none of its basis elements are contained in compact superspaces. Take an arbitrary basic, open set $U = (\prod_{k=1}^{n} [a_k, b_k]) \times \mathbb{R} \times \cdots$. If U were contained in a compact set, then its closure $\overline{U}$ would be compact as well, since closed subsets of compact spaces are compact. However, $\overline{U}$ is the finite product of closed intervals, which is also multiplied by infinitely many instances of $\mathbb{R}$, making it not compct.

Of course, there are weird and arcane topological spaces. One method of demystifying such spaces is by considering them as subspaces of nice topological spaces, in hopes that it will inherit some nice properties. For a "bad" space X, we can consider an embedding $f : X \to Y$ into a "good" space Y, such that $(X \cong f(X)) \subseteq Y$. In reality, $f(X)$ is the true subspace of Y, but since X is homeomorphic to $f(X)$, X has the nice properties $f(X)$ would inherit as a subspace of Y. Through this, we are able to study a bad space by observing it as a subspace of a good space.

So what constitutes a good space? The examples we will consider are metrizable spaces and compact Hausdorff spaces. Metrizable spaces are nice due to a rigid topology defined by their metric, as well as topological notions (such as limit points and closure) being able to be probed by sequences. Compact Hausdorff spaces are nice since infinite covers can be reduced to finite subcovers, so some properties are more tractable, as well as sequences having unique limit points and other goodies.

Subspaces of metrizable spaces are metrizable as well, so no new information can be gathered from this. While subspaces of compact Hausdorff spaces are Hausdorff, they do not need to be compact. For example, consider the compact Hausdorff space $[0, 1]$ and a subspace $(0, 1)$, which is obviously not compact. In later sections we will discuss a more complete characterization of *every* space that can be embedded into a compact Hausdorff space, but for now we investigate a simpler question. We investigate what classes of spaces are such that adding one specific point can produce a compact Hausdorff space. We will see that it is indeed locally compact Hausdorff spaces: there is a special point one could adjoin to a locally compact Hausdorff space to produce a compact Hausdorff space. This process is called **one-point compactification**.

[3.7.3] Theorem (*Characterization of Locally Compact Hausdorff Spaces*)

Let X be a space. Then X is locally compact Hausdorff if and only if there exists a space Y such that:

(1) X is a subspace of Y.

(2) $Y \setminus X$ is a singleton.

(3) Y is compact Hausdorff.

The identified space Y is unique up to homeomorphism. If there are two spaces Y, Y' satisfying the aforementioned properties, then there is a homeomorphism from Y to Y' that fixes X.

Proof. We first verify uniqueness. Let X be a LCH space, and suppose Y, Y' are superspaces of X that have the prescribed properties, in such a way that $Y = X \cup \{\infty\}$ and $Y' = X \cup \{\infty'\}$. Define a map

$$h : Y \to Y' \quad h(x) = \begin{cases} \infty' & x = \infty, \\ x & \text{otherwise,} \end{cases}$$

a map that fixes X and takes $\infty \mapsto \infty'$. We show that h is a homeomorphism.

Obviously, h is a bijection. We show that h maps open sets to open sets, and the reverse is true by symmetry. To get started, suppose $O \subseteq Y$ is open. If $\infty \notin O$, then $O \subseteq X$ is open, and so $h(O) = O \subseteq X$ is also open. If $\infty \in O$, then note that $C = Y \setminus O$ is a closed subset of X, and so C is compact (since closed subspaces of Hausdorff spaces are compact). Since $X \subseteq Y'$, we have that C is a compact, and thus closed, subspace of Y'. Thus $h(O) = Y' \setminus C$ is open, making h a homeomorphism.

Now we show that if X is a LCH space, there is some point $\infty \notin X$ such that $Y = X \cup \{\infty\}$ is compact Hausdorff. We equip Y with the topology comprised of all sets U that are open subsets of X and all sets $Y \setminus C$ such that C is compact in X. We need to show that this topology is valid, the topology on X is the subspace topology from Y, and that Y is compact Hausdorff with this topology. The first two conditions are trivial to check.

We show Y is compact: consider an open cover $\mathscr{A}$ of Y. Note that there is some compact $C \subseteq X$ such that $Y \setminus C \in \mathscr{A}$, since open sets in X do not contain ∞. Then note all other members of $\mathscr{A}$, when intersected with A, form an open cover for C with open subsets of X. Since C is compact, a finite subcollection covers C, and when adjoined to the open set $Y \setminus C$, a finite subcollection of $\mathscr{A}$ covers Y. Thus Y is compact.

We show Y is Hausdorff; consider $x \neq y \in Y$. Separated neighborhoods exist if $x, y \in X$ since X is Hausdorff, so WLOG suppose $y = \infty$. Since X is locally compact, there is a compact $C \subseteq X$ such that there is some open neighborhood $U \subseteq C$ of x. Then $U, Y \setminus C$ are disjoint open neighborhoods of x and y respectively, and so Y is Hausdorff.

Now we prove the converse: if such a space Y exists that satisfies the following conditions, then X is locally compact (Hausdorffness is immediate). Given $x \in X$, we show X is locally compact at x. Choose disjoint open sets U, V of Y containing x and ∞. Then $C = Y \setminus V$ is closed in Hausdorff space Y and thus compact, and U is an open neighborhood of x that lies in the compact space $C \subseteq X$. Thus X is locally compact. ❦

If X happens to be a compact space, then the $Y = X \cup \{\infty\}$ from the preceding theorem is not interesting. Note if X is compact, we need that $Y \setminus X$ is open in our prescribed Y, so the singleton $\{\infty\}$ is open, making ∞ an isolated point. If X is not compact, then $Y \setminus X$ is not open, and so $\{\infty\}$ is not open. Thus since every open neighborhood of ∞ intersects X nontrivially, making ∞ a limit point of X. Thus $\overline{X} = Y$ for X not compact, where the closure is relative to Y. We call this process **compactification**.

[3.7.4] Definition (*Compactification, One-Point Compactification*)

If Y is a compact Hausdorff space and X is a proper subspace of Y such that $\overline{X} = Y$, Y is called the **compactification** of X. If $Y \setminus X$ is a singleton, then it is called the **one-point compactification**.

[3.7.5] Example (*Common Examples of Compactifications*)

Ultimately, we have shown that if X is LCH and not compact, then it admits a one-point compactification. Note this means that $\mathbb{R}$ admits the one-point compactification $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$, which can be readily seen to be homeomorphic to $\mathbb{R}^1$ by stereographic projection. The same is true for $\mathbb{R}^2$ and in general $\mathbb{R}^n$ with S^n. Note if we see $\mathbb{C} \cong \mathbb{R}^2$, we get a one-point compactification $\mathbb{C} \cup \{\infty\} \cong S^2$, and we even call the one-point compactification of $\mathbb{C}$ the *Riemann sphere*.

Note there is an alternative, more satisfying way to define local compactness. Typically, when a property is local about some $x \in X$, it means that there exists a neighborhood about x where the property is true. But our definition of local compactness is not framed in this way. Indeed, we will formulate an alternative definition that identifies behavior more local in nature. Note this characterization is only equivalent in the context of Hausdorff spaces, however.

[3.7.6] Theorem (*Characterization of Local Compactness*)

Let X be a Hausdorff space. Then X is locally compact if and only if, for any $x \in X$ and an open neighborhood U of x, there is some open neighborhood $V \subseteq U$ of x such that $\overline{V}$ is compact and $\overline{V} \subseteq U$.

Proof.

($\Longrightarrow$): Suppose X is LCH. Let $x \in X$ and an open neighborhood $U \subseteq X$ of x be arbitrary. Let Y be the one-point compactification of X, then note $C = Y \setminus U$ is closed, and thus compact, in the Hausdorff space Y. Note that a compact subset of a Hausdorff space can be separated from a singleton outside the compact subset through open sets. Thus choose disjoint open sets V, W such that $x \in V$ and $C \subseteq W$. Thus $\overline{V}$ is closed and thus compact in the Hausdorff space Y, and since it is disjoint from C, we have $\overline{V} \subseteq U$ as desired.

($\Longleftarrow$): Suppose this formulation is true. Given $x \in X$ and an open neighborhood $U \subseteq X$, there exists a neighborhood V of x such that the compact space $\overline{V}$ is contained in U. Since $\overline{V}$ is a compact subspace containing the open neighborhood V of x, it follows that X is locally compact.

That is, a Hausdorff space X is locally compact if and only if every neighborhood about any point $x \in X$ contains a compact subspace that contains an open neighborhood about x.

[3.7.7] Corollary (*Locally Compact Subspaces of LCH Spaces*)

Let X be an LCH space. If $A \subseteq X$ is open or closed, then A is locally compact.

> **Proof.** Suppose A is closed, and let $x \in A$. Note that X is locally compact, so there is some compact set $C \subseteq X$ that contains an open neighborhood $U \subseteq X$ of x. Then note $C \cap A$ is closed in C and thus compact, and it contains the open neighborhood $U \cap A$ of x in A.
>
> Now suppose A is open, and let $x \in A$. Since X is locally compact and A is a neighborhood of x, there is some neighborhood $V \subseteq A$ of x such that $\overline{V} \subseteq A$ is compact. Then $\overline{V}$ is a compact space containing the open neighborhood V of x, and so A is locally compact. ❦

[3.7.8] Theorem (*Characterization of Open Subspaces of Compact Hausdorff Space*)

A space X is homeomorphic to an open subspace of a compact Hausdorff space if and only if X is locally compact Hausdorff.

3.8 Nets

[3.8.1] Definition (*Partially Ordered Set, Directed Set*)

A **partially ordered set** is a set A together with a partial order $\leq$ satisfying the following conditions.

(1) **Reflexivity.** $a \leq a$ for every $a \in A$.

(2) **Symmetry.** For every pair $a, b \in A$, if $a \leq b$ and $b \leq a$, then $a = b$.

(3) **Transitivty.** For every triple $a, b, c \in A$, if $a \leq b$ and $b \leq c$, then $a \leq c$.

A **directed set** is some partially ordered set Λ such that for any pair $a, b \in \Lambda$, there is some $c \in \Lambda$ such that $a \leq c$ and $b \leq c$. That is, any pair of elements in Λ have an upper bound in the set.

[3.8.2] Definition (*Cofinal Sets*)

A subset K of a directed set Λ is **cofinal** if, for any $a \in \Lambda$, there is some $b \in K$ such that $a \leq b$.

[3.8.3] Remark (*Intuition behind Cofinality*)

Cofinality is precisely the algebraic translation of "marching off to infinity." For a standard sequence indexed by $\mathbb{Z}_+$, a subset of indices goes to infinity because it eventually exceeds any given integer N. For an arbitrary directed set Λ, a subset is cofinal if it eventually exceeds any given element $\alpha \in \Lambda$.

[3.8.4] Theorem

Cofinal subsets of a directed set are directed sets.

Proof. Let K be a cofinal subset of the directed set Λ. For any $a, b \in K$, note there is some $c \in \Lambda$ such that $a \leq c$ and $b \leq c$. Since K is cofinal, there is some $k \in K$ such that $c \leq k$. Thus $a, b \leq k$, and so K is a directed set. ❦

[3.8.5] Definition (*Net*)

Let X be a topological space. A **net** in X is a tuple of X indexed by Λ. That is, a net $\mathbf{x}$ is a map $f : \Lambda \to X$ such that $\mathbf{x} = (x_\lambda)_{\lambda \in \Lambda}$, where $x_\lambda = f(\lambda)$ for each $\lambda \in \Lambda$.

[3.8.6] Definition (*Convergence of Net*)

A net $(x_\lambda)_{\lambda \in \Lambda}$ is said to **converge** to a point x if, for any open neighborhood U of x, there is some $\lambda \in \Lambda$ such that $x_{\lambda'} \in U$ for every $\lambda' \geq \lambda$.

Many theorems about sequences can be lifted to nets in a completely identical way. We repeat them here anyways.

[3.8.7] Theorem (*Hausdorff Guarantees Unique Limit Point*)

Suppose X is a Hausdorff space. Then a convergent net $\mathbf{x} = (x_\lambda)_{\lambda \in \Lambda}$ has a unique limit.

Proof. For the sake of contradiction, suppose $\mathbf{x} \to x$ and $\mathbf{x} \to y$ for $x \neq y \in X$. Construct disjoint open neighborhoods U_x and U_y of x and y respectively. Then note the tail of $\mathbf{x}$ cannot lie in U_x and U_y simultaneously, since they are disjoint, which is a contradiction. Thus nets converge to, at most, one value.

[3.8.8] Theorem (*Nets Probe Elements of Closure*)

Let X be a topological space with $A \subseteq X$. Then $x \in \overline{A}$ if and only if a net of points from A converges to x.

Proof.

($\Longrightarrow$): Suppose $x \in \overline{A}$. Let Λ be the collection of all open neighborhoods of x, directed by reverse inclusion so that $U_\lambda \subseteq U_{\lambda'}$ for every $\lambda \geq \lambda'$ (where $U_\lambda = \lambda$). Since $x \in \overline{A}$, there is some $x_\lambda \in U_\lambda \cap A$ for each $\lambda \in \Lambda$. Define the net $\mathbf{x} = (x_\lambda)_{\lambda \in \Lambda}$. For any open neighborhood U of x, note that there is some $\lambda \in \Lambda$ such that $U_\lambda \subseteq U$ (specifically, choosing $\lambda = U$). Thus, for each $\lambda' \geq \lambda$, we have that $x_{\lambda'} \in U_{\lambda'} \subseteq U$, meaning the net $\mathbf{x}$ converges to x.

($\Longleftarrow$): Suppose $\mathbf{x} = (x_\lambda)_{\lambda \in \Lambda}$ is a net of elements in A that converges to $x \in X$. For any open neighborhood U of x, note there is some $\lambda \in \Lambda$ such that $x_\lambda \in U$. Since $x_\lambda \in A$, U intersects A, meaning $x \in \overline{A}$.

[3.8.9] Theorem (*Nets Probe Continuity*)

Let $f : X \to Y$. Then f is continuous if and only if, for every convergent net $\mathbf{x} = (x_\lambda)_{\lambda \in \Lambda} \to x$, the net $f(x_\lambda) \to f(x)$.

Proof.

($\Longrightarrow$): Suppose f is continuous. Let V be any open neighborhood of $f(x)$. By continuity, note that $U = f^{-1}(V)$ is an open neighborhood of x such that $f(U) \subseteq V$. Then there is some $\lambda' \in \Lambda$ such that $x_\lambda \in U$ for each $\lambda \geq \lambda'$. Thus $f(x_\lambda) \in f(U) \subseteq V$ for every $\lambda \geq \lambda'$, meaning $f(x_\lambda) \to f(x)$.

($\Longleftarrow$): Let A be any subset of X. For any $x \in \overline{A}$, note there is some convergent net $\mathbf{x} = (x_\lambda)_{\lambda \in \Lambda}$ drawing values from A such that $\mathbf{x} \to x$. By hypothesis, note $f(\mathbf{x}) \to f(x) \in \overline{f(A)}$. Thus $f(\overline{A}) \subseteq \overline{f(A)}$, completing the proof.

Nets seem nice so far, but the true problem is that subsequences don't generalize well. If we generalized subsequences naively by just considering a subset of indices, subnets would be too weak to do anything.

[3.8.10] Remark (*The Failure of Naive Subnets*)

Suppose we tried to define a subnet by restricting f to a cofinal subset $J' \subseteq J$, in exact analogy with passing to a subsequence. Taking $J = \mathbb{Z}_+$, this recovers precisely the notion of a subsequence, so any theorem we prove about such "subnets" would specialize to a theorem about subsequences. But we already know that subsequences fail to detect compactness: the space $[0,1]^{[0,1]}$ is compact by Tychonoff, and yet it contains a sequence with no convergent subsequence. Under the naive definition, then, this compact space would admit a net with no convergent subnet, and the characterization of compactness we are after would be false.

The obstruction is one of size. To converge in a product, a net must eventually settle in every coordinate at once, and here there are uncountably many coordinates to satisfy simultaneously. A cofinal subset of $\mathbb{Z}_+$ is still countable, so no matter how cleverly we thin the indices, we have only countably many choices with which to meet uncountably many demands. Thinning cannot help us.

The repair, then, is not to select *fewer* indices but to permit *more*. By allowing an entirely new directed set K together with a map $g : K \to J$, we free ourselves from the size of J altogether: K may be far larger than J, and g need not be injective, so the subnet is permitted to revisit indices and to be indexed by a set of much greater cardinality than the original net. What remains is to insist that K still traverses J faithfully, and this is exactly what monotonicity and cofinality accomplish.

[3.8.11] Definition (*Subnet*)

Let $f : J \to X$ be a net relative to some directed set J. If K is a directed set together with some map $g : K \to J$ such that

(1) For any $i, j \in K$ such that $i \leq j$, we have $g(i) \leq g(j)$ (monotonicity),

(2) $g(K)$ is cofinal in J,

then we say the map $f \circ g : K \to X$ is a **subnet** of f.

These properties allow the subnet $f \circ g$ relative to K to lift properties of subsequences to subnets: monotonicity ensures we do not jump backward in the net, and cofinality ensures the subnet marches arbitrarily far out into the original directed set J.

[3.8.12] Theorem (*Convergence of Subnets*)

If a net converges to a point, then all subnets must converge to the same point.

[3.8.13] Definition (*Accumulation Point*)

Let $(x_\lambda)_{\lambda \in \Lambda}$ be a net in X. We say that x is an **accumulation point** of the net if, for any open neighborhood U of x, the set of indices λ such that $x_\lambda \in U$ is cofinal in Λ.

[3.8.14] Remark (*Cofinality and Accumulation Points*)

For sequences, an accumulation point is one that the sequence visits "infinitely often." Cofinality is the exact topological translation of "infinitely often" for nets. If x is an accumulation point, then no matter how far out we march into the directed set Λ, we can always find an index further out where the net jumps back into U. Because this set of "good" indices is cofinal, we can build a new directed set K that maps entirely into them, creating a subnet that stays inside U permanently (i.e., a subnet that converges to x).

[3.8.15] Theorem (*Accumulation Points are Limit Points of Subnet*)

A net has an accumulation point if and only if a subnet converges to the accumulation point.

[3.8.16] Theorem (*Equivalence of Compactness and Sequential Compactness via Nets*)

A space X is compact if and only if every net in X has a convergent subnet.

The aforementioned theorem says that compact spaces are sequentially compact relative to nets (i.e., if every net has an accumulation point). This closely mirrors the equivalence of compactness and sequential compactness in metric spaces.

4 Countability and Separation Axioms

4.1 The Countability Axioms

[4.1.1] Definition (*Countable Base, First Countable Space*)

Suppose $(X, \mathcal{T})$ is a topological space. We say some $x \in X$ admits a **countable basis** at x if there exists some countable collection $\mathcal{B}$ of open neighborhoods of x such that, for every open neighborhood U of x, U contains some element of $\mathcal{B}$. If all points in X satisfy this property, we say that X satisfies the **first-countability axiom**, or is **first-countable**.

[4.1.2] Example (*Metrizable Spaces are First-Countable*)

Suppose X is a metrizable topological space. For any $x \in X$, consider the collection of open neighborhoods $\mathcal{B} = \{B_n\}_{n \in \mathbb{Z}_+}$ about x such that $B_n = B(x, 1/n)$. Then, for any basic open neighborhood $U = B(x, \varepsilon)$ about x, we have that there is some $n \in \mathbb{Z}_+$ such that $1/n < \varepsilon$, and so $B_n(x, 1/n) \subseteq U$, making $\mathcal{B}$ a countable basis. Thus X is first-countable.

First countability is a property that lets convergent sequences probe limit points and the continuity of functions.

[4.1.3] Theorem

Let X be a topological space.

(1) Suppose $A \subseteq X$. If $(x_n) \subseteq A$ is a convergent sequence such that $x_n \to x$, then $x \in \overline{A}$. The converse is true if X is first-countable.

(2) If f is a continuous function, then for any $x_n \to x$, we have that $f(x_n) \to f(x)$. The converse is true if X is first-countable.

Proof.

(1) $(\Longrightarrow)$: This follows immediately by the definition of convergence of a sequence in a topological space.

$(\Longleftarrow)$: Suppose $x \in \overline{A}$. Note x admits some countable basis $\mathcal{B} = \{B_n\}_{n \in J}$. We can make a new countable basis from this, say $\mathcal{U} = \{U_n\}_{n \in J}$ such that $U_n = B_1 \cap \cdots \cap B_n$. Of course, each U_k is open and $x \in U_k$, so each U_k is an open neighborhood of x. Since $x \in \overline{A}$, every open neighborhood of x intersects A. Thus, for each $k \in J$, note that U_k is an open neighborhood of x, so we are justified in choosing x_k such that $x_k \in (U_k \cap A)$. If J is countably infinite, then $(x_n) \subseteq A$ is a sequence such that $x_n \to x$, obviously. If J is finite, let $n = \max(J)$. There exists some $k \in J$ such that, for any open neighborhood O of x, we have that $U_n \subseteq U_k \subseteq O$, with the first inclusion due to $\mathcal{U}$ defining a nested countable basis at x. Upgrade the list $(x_1, \ldots, x_n)$ to the sequence $(x_1, \ldots, x_n, x_n, x_n, \ldots)$, and note that $x_n \to x$.

(2) $(\Longrightarrow)$: Suppose f is continuous, and let $(x_n) \subseteq X$ be a sequence such that $x_n \to x$. Let V be some open neighborhood of $f(x)$, and by continuity of f, there exists an open neighborhood U of x such that $f(U) \subseteq V$. Note that there is some $N \in \mathbb{Z}_+$ such that $x_n \in U$ for every $n \geq N$, and since $f(U) \subseteq V$, we have that $f(x_n) \in V$ for every $n \geq N$, and so $f(x_n) \to f(x)$.

$(\Longleftarrow)$: We show f is continuous by showing that, for any $A \subseteq X$, we have that $f(\overline{A}) \subseteq \overline{f(A)}$. For any $x \in \overline{A}$, note that there is some sequence $(x_n) \subseteq A$ such that $x_n \to x$ by (1). By hypothesis, we have that $f(x_n) \to f(x)$, and we also have that $f(x) \in \overline{f(A)}$ from (1). Thus $f(\overline{A}) \subseteq \overline{f(A)}$, and so f is continuous.

❦

[4.1.4] Definition (*Second-Countability*)

If a topological space X admits a countable basis for its topology, then we say X satisfies the **second-countability axiom**, or is **second countable**.

Obviously, if X is second-countable, then it is first-countable, since every open neighborhood U of some $x \in X$ can be written as the union of a subset of a countable basis from X, and so x admits a countable basis.

[4.1.5] Example (*Examples of Second-Countable Spaces*)

Note that $\mathbb{R}$ admits a countable basis, where each element takes th eform (a,b) for rational endpoints $a,b \in \mathbb{Q}$. Similarly, $\mathbb{R}^n$ admits a countable basis where each element takes the form $\prod_{k=1}^n (a_k, b_k)$, where $a_k, b_k \in \mathbb{Q}$ for every k. Even $\mathbb{R}^\omega$ in its product topology admits a countable basis, with each set taking the form $\prod_{k=1}^n (a_k, b_k) \times \prod_{k \in \mathbb{Z}_+ \setminus \{1,\dots,n\}} \mathbb{R}$ for $a_k, b_k \in \mathbb{Q}$ for each k.

The uniform topology on $\mathbb{R}^\omega$ is metrizable by the uniform metric $\overline{\rho}(x,y) = \sup_{n \in \mathbb{Z}_+} \{\overline{d}(x_n, y_n)\}$, where $\overline{d}(x_n, y_n) = \min(|x_n - y_n|, 1)$; hence, it is first-countable.

To show $\mathbb{R}^\omega$ is not second-countable, we appeal to the lemma that if a space X has a countable basis $\mathcal{B}$, then any discrete subspace $A \subseteq X$ is countable. Indeed, since each $\{a\}$ is open in A, there is some open set $U \subseteq X$ such that $U \cap A = \{a\}$. By the definition of a basis, there exists some $B_a \in \mathcal{B}$ with $a \in B_a \subseteq U$, yielding $B_a \cap A = \{a\}$. For any distinct $a, b \in A$, the basis elements B_a and B_b are not equal since $a \in B_a$ but $a \notin B_b$ (as $B_b \cap A = \{b\}$). The map $a \mapsto B_a$ is therefore an injection of A into $\mathcal{B}$, proving A must be countable.

Finally, consider the subspace $A \subseteq \mathbb{R}^\omega$ comprising all sequences of 0s and 1s. This set A is uncountable. However, for any distinct $a, b \in A$, they must differ in at least one coordinate, so $\overline{\rho}(a,b) = 1$. The subspace A therefore inherits the discrete metric, giving it the discrete topology. By contrapositive, $\mathbb{R}^\omega$ is not second-countable.

[4.1.6] Theorem (*First/Second-Countability Preserved Under Subspace and Product Topologies*)

A subspace of first/second-countable spaces is first/second-countable. The countable product of first/second-countable spaces is first/second-countable.

Proof. The proof is entirely mechanical and a straightforward application of the definition. ❦

[4.1.7] Definition (*Density*)

A set $A \subseteq X$ is said to be **dense** if $\overline{A} = X$.

[4.1.8] Definition (*Lindelöf Space*)

A space X is said to be **Lindelöf** if every open cover of X admits a countable subcover.

[4.1.9] Definition (*Separable Space*)

A space X is said to be **separable** if there exists a dense subset of X that is countable.

[4.1.10] Theorem

Suppose that X is second-countable.

(1) X is Lindelöf.

(2) X is separable.

That is, second-countability implies being Lindelöf and separable.

Proof. Let $\mathcal{B} = \{B_n\}$ be a countable basis for X.

(1) Let $\mathcal{A}$ be a cover for X. For each $x \in X$, there is some $A \in \mathcal{A}$ such that $x \in A$. Since A is open, there is a basic, open neighborhood B_n such that $x \in B_n \subseteq A$. Choose exactly one $A \in \mathcal{A}$ such that $B_n \subseteq A$, and let $A_n = A$. Then $\{A_n\}$ covers X, forming a countable subcover from $\mathcal{A}$.

(2) For each $n \in \mathbb{Z}_+$ such that $B_n \neq \emptyset$, choose some $x_n \in B_n$, and let D be a set comprising each x_n. Then note that for any basic open neighborhood B_k about any $x \in X$, $x_k \in B_k \cap D$, and so $\overline{D} = X$. Thus D is dense in X.

[4.1.11] Example (*Countability Properties of Sorgenfrey Line*)

Consider the set $\mathbb{R}_\ell$ endowed with the lower limit topology, also known as the *Sorgenfrey line*. We will show that it is first-countable, separable, and Lindelöf, but not second-countable.

Of course, any $x \in \mathbb{R}_\ell$ admits a countable basis $\{[x, x+1/n)\}_{n\in\mathbb{Z}_+}$, so $\mathbb{R}_\ell$ is first-countable. It is also easy to see that $\mathbb{Q}$ is dense in $\mathbb{R}_\ell$, and so $\mathbb{R}_\ell$ is separable.

We show $\mathbb{R}_\ell$ is not second-countable. Suppose $\mathcal{B}$ is a basis for the lower limit topology. Then, for any $x \in X$, let B_x be some basis element such that $x \in B_x \subseteq [x, x+1)$. For any $x \neq y \in \mathbb{R}_\ell$, note that $B_x \neq B_y$, since $[x = \min(B_x)] \neq [y = \min(B_y)]$. Thus the map from $\mathbb{R}_\ell$ to $\mathcal{B}$, defined by $x \mapsto B_x$, is injective, and so $|\mathbb{R}_\ell| < |\mathcal{B}|$, making the basis uncountable.

The proof of $\mathbb{R}_\ell$ being Lindelöf is annoying.

[4.1.12] Example (*Product of Lindelöf Spaces is not Lindelöf*)

Lindelöf-ness is not a property that is preserved under products. For example, even though $\mathbb{R}_\ell$ is Lindelöf, we can quickly show $\mathbb{R}_\ell \times \mathbb{R}_\ell$ (also known as $\mathbb{R}_\ell^2$, the *Sorgenfrey plane*) is not. Choose the basis that has sets of the form $[a,b) \times [c,d)$, and consider the set

$$L = \{(x, -x) : x \in \mathbb{R}_\ell\},$$

which is essentially the graph of $f(x) = -x$ in $\mathbb{R}_\ell^2$. It is quick to show L is closed, so we have that $\mathbb{R}_\ell^2 \setminus L$ is open. Then consider the open cover formed by $\mathbb{R}_\ell^2 \setminus L$ adjoined with open sets of the form $[a,b) \times [-a,c)$. Every basis element (except the first) intersects L, which is uncountable, and so there is no countable subcover. Thus $\mathbb{R}_\ell^2$ is not Lindelöf.

[4.1.13] Example (*Subspace of Lindelöf Space is not Lindelöf*)

The ordered square I_o^2 is compact (and thus Lindelöf), but the subspace $I \times (0,1)$ is not Lindelöf.

[4.1.14] Recap

(1) A space X is **first-countable** at a point $x \in X$ (i.e., satisfies the **first-countability axiom** at x) if there exists a countable collection of open neighborhoods $\{B_n\}$ of x such that, for any open neighborhood U of x, there is some $n \in \mathbb{Z}_+$ for which $B_n \subseteq U$.

(2) From any countable basis $\{B_n\}$ at a point $x \in X$, one can derive a countable basis that is "nested" (i.e., $U_1 \supseteq U_2 \supseteq \cdots$). Indeed, let $U_n = B_1 \cap \cdots \cap B_n$. Then note x is in each B_k, and so x is in each U_k. Moreover, the finite intersection of open sets is open, so each U_k is an open neighborhood of x. Obviously $\{U_n\}$ forms a nested chain of open neighborhoods of x.

(3) First countability is a property that lets convergent sequences probe limit points and the continuity of functions. For a space X and some $A \subseteq X$, if we have a sequence $(x_n) \subseteq X$ such that $x_n \to x$, then $x \in \overline{A}$ by definition of convergence in a topological space. The converse is not necessarily true in a general topological space, but if X is first-countable, then it is immediate (simply take a countable basis at any $x \in X$, make it nested, then choose a sequence as one would in a metric space).

(4) Similarly, if $f : X \to Y$ is continuous, then for any $(x_n) \subseteq X$ such that $x_n \to x$, we have that $f(x_n) \to f(x)$ (simply by the property that if V is an open neighborhood of $f(x) \in f(X)$ for continuous f, there is some open neighborhood U of x such that $f(U) \subseteq V$). The converse is only true in a first-countable space, however. Using the previous theorem about sequences, one can prove that $f(\overline{A}) \subseteq \overline{f(A)}$.

(5) A space X is **second-countable** if it admits a countable basis.

(6) A subset $D \subseteq X$ is **dense** if $\overline{D} = X$.

(7) A space X is **Lindelöf** if every open covering of X admits a countable subcovering.

(8) A space X is **separable** if there exists a dense subset of X that is countable.

(9) Second-countability of a space X implies the space being Lindelöf and separable. If X is second-countable, it admits a countable basis $\{B_n\}_{n \in J}$. For any open covering $\mathscr{A}$ of X, note that for any $x \in X$, there is some $A \in \mathscr{A}$ such that $x \in A$. Since A is open, there is a basis element B_n such that $x \in B_n \subseteq A$. For all $x \in B_n$, choose this representative A again when querying the cover for a member that contains an element. Associating A with n by A_n, note that $\{A_n\}$ is an open subcover of X, making X Lindelöf. Furthermore, for each $n \in J$, choose some $x_n \in B_n$, and let D be the countable set of all x_n. For any $x \in X$, any basic open neighborhood intersects D (since it contains elements from every basic, open set), and so $\overline{D} = X$, making X separable as well.

4.1.1 Exercises

[4.1.15] Problem

For a topological space X, we say a subset $A \subseteq X$ is G_δ if A can be written as the countable intersection of open sets. Show that singletons are G_δ space in first-countable, T_1 topological spaces.

Proof. Let X be first-countable and T_1, and suppose $\{x\} \subseteq X$. Let $\{B_n\}_{n\in J}$ be a countable basis about x: we will show $\bigcap_{n\in J} B_n = \{x\}$. Simply, if $y \in \bigcap_{n\in J} B_n$, then y is in every open neighborhood of x, making y a limit point of $\{x\}$. But $\{x\}$ is closed in a T_1 space, menaing it contains all its limit points, and so y is forced to be x, completing the proof.

[4.1.16] Problem

Show every compact metrizable space X has a countable basis.

Proof. For any $n \in \mathbb{Z}_+$, note the set of all balls of radius $1/n$ forms an open cover for X, and by compactness we have a finite subcover $\mathcal{A}_n = \{A_n^i\}_{i=1}^k$. Then consider $\mathcal{B} = \bigcup_{n\in\mathbb{Z}_+} \mathcal{A}_n$, a countable set of open balls that cover X. To show $\mathcal{B}$ is a basis, we show that for any open neighborhood U of x, there is some $B \in \mathcal{B}$ such that $x \in B \subseteq U$.

Since U is open, there is some $\varepsilon > 0$ such that $B(x,\varepsilon) \subseteq U$. Choose some $n \in \mathbb{Z}_+$ such that $2/n < \varepsilon$. Since $\mathcal{A}_n$ covers X, there is some ball $B = B(p, 1/n) \in \mathcal{A}_n$ such that $x \in B$. Then note, for any $z \in B$, we have that

$$d(z,x) \leq d(z,p) + d(p,x) < \frac{1}{n} + \frac{1}{n} = \frac{2}{n} < \varepsilon,$$

and so $x \in B \subseteq B(x,\varepsilon) \subseteq U$, completing the proof.

[4.1.17] Problem

Show that if a separable space is metrizable, then it is second-countable.

Proof. Suppose (X,d) has a countable subset $D = \{x_k\}_{k\in J}$, where J is countable, such that $\overline{D} = X$. For each $k \in J$, let $\mathcal{B}_k = \{B(x_k, 1/n)\}_{n\in\mathbb{Z}_+}$. Then note $\mathcal{B} = \bigcup_{k\in J} \mathcal{B}_k$ is a countable, open cover of X—we show that $\mathcal{B}$ forms a basis.

Let U be some open neighborhood of x, and let $\varepsilon > 0$ be such that $x \in B(x,\varepsilon) \subseteq U$. Since $\overline{D} = X$, any open neighborhood of x intersects some $x_k \in D$. Let $n \in \mathbb{Z}_+$ be such that $x_k \in B(x, 1/n) \cap D$, and also be such that $2/n < \varepsilon$. Thus $d(x,x_k) < 1/n$, and so for any $z \in B(x_k, 1/n)$, we have that

$$d(z,x) \leq d(z,x_k) + d(x_k,x) < 2/n < \varepsilon,$$

and so $x \in \underbrace{B(x_k, 1/n)}_{\in\mathcal{B}} \subseteq B(x,\varepsilon) \subseteq U$.

[4.1.18] Problem

Show that if a Lindelöf space X is metrizable by a metric d, then it is second-countable.

Proof. For each $n \in \mathbb{Z}_+$, let $\mathcal{A}_n$ be the set of all open balls of radius $1/n$, which forms an open cover for X. Since X is Lindelöf, there is a countable subcover A_n. We conjecture that $\mathcal{B} = \bigcup_{n \in \mathbb{Z}_+} A_n$ is a countable basis for the metric topology of X. Of course, it is countable, and it covers X. We now show it has the basis property.

Let $x \in X$ be arbitrary, and let U be any open neighborhood of x. Since U is open, there is some $\varepsilon > 0$ such that $B(x, \varepsilon) \subseteq U$. Let $n \in \mathbb{Z}_+$ be such that $2/n < \varepsilon$. Note that $\mathcal{B}$ covers X with balls of radius $1/n$, and so there is some $B(y, 1/n) \in \mathcal{B}$ that is an open neighborhood for x. Then, for any $z \in B(y, 1/n)$, note that

$$d(z,x) \leq d(z,y) + d(y,x) < 2/n < \varepsilon,$$

and so $x \in \underbrace{B(y, 1/n)}_{\in \mathcal{B}} \subseteq B(x, \varepsilon) \subseteq U$. ❦

[4.1.19] Remark

We have shown that, in metrizable spaces, being Lindelöf, separable, and second-countable are all equivalent.

[4.1.20] Problem

Show that if X is separable, then every collection of disjoint, open sets in X is countable.

Proof. Let $D = \{x_n\}_{n \in J}$ be a countable subset of X such that $\overline{D} = X$ (i.e., D is dense in X). Let $\mathcal{A}$ be a collection of disjoint, open sets in X. Without loss of generality, suppose $\mathcal{A}$ does not contain the empty set. Since each $A \in \mathcal{A}$ is an open neighborhood of some element of X, there is some $k \in J$ for which $x_k \in A$ by the density of D in X. Since each $A \in \mathcal{A}$ is disjoint, the map from $\mathcal{A}$ to D given by $A \mapsto x_k$ is injective. Thus $|\mathcal{A}| \leq |D|$, making $\mathcal{A}$ countable. ❦

4.2 Separation Axioms

[4.2.1] Definition (*Topologically Distinguishability, Separation*)

Let x, y be two points in a topological space X. Then x, y are **topologically indistinguishable** if the set of all open neighborhoods about x coincides with the set of all open neighborhoods about y. Thus, x, y are **topologically distinguishable** if one point has an open neighborhood that excludes the other. Moreover, x, y are **separated** if there exist open neighborhoods about x and y that are disjoint. Separation generalizes easily to a point and a set or two sets.

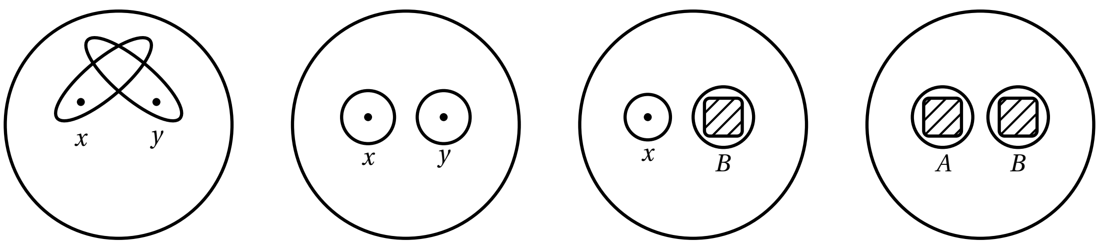
Note that when dealing with connected spaces, we also use separations, but only separations whose union is the entire space. In dealing with separation axioms, we still deal with separations (i.e., disjoint, open sets), but we do not care if their union is the entire space.

[4.2.2] Definition (*Regular, Normal Spaces*)

Suppose a space X is T_1 (i.e., all singletons are closed). We say that X is **regular** if, for any point x and a closed set $B \subseteq X$ disjoint from x, we may separate x and B. We say that X is **normal** if any pair of disjoint, closed sets can be separated.

Note that we specify B to be closed in the previous definition since, if $B \subseteq X$ is not closed, then we would be unable to separate a point $x \in \overline{B} \setminus B$ from B, making most spaces X not regular. The same is true for the definition of normality (consider a set A and a singleton set B that contains a point from $\overline{A} \setminus A$).

Obviously, a regular space is Hausdorff (consider a point x and the closed space $\{y\}$), and analogously, normal spaces are regular.



T_1 Hausdorff Regular Normal

[4.2.3] Theorem (*Characterizations of Regularity and Normality*)

Suppose X is T_1.

(1) X is regular if and only if, for any $x \in X$ with any open neighborhood U of x, there is some open neighborhood V of x such that $\overline{V} \subseteq U$.

(2) **Shrinking Lemma.** X is normal if and only if, for any closed set $A \subseteq X$ contained in any open neighborhood U, there is some open set V that contains A such that $\overline{V} \subseteq U$.

Proof.

(1) ($\Longrightarrow$): Suppose X is regular. Consider a point $x \in X$ and some open neighborhood U of x. Set $B = X \setminus U$, a closed set; by regularity of X, there exist disjoint, open neighborhoods V, W of x and B. If we show that $\overline{V} \cap B = \emptyset$, then we have shown that $\overline{V} \subseteq U$. Indeed, if $y \in B$, then there is an open neighborhood W of y that does not intersect V, and so $y \notin \overline{V}$. Thus $\overline{V} \subseteq U$.

($\Longleftarrow$): Let $x \in X$ be a point disjoint from some closed set $B \subseteq X$. Then note $U = X \setminus B$ is an open neighborhood of x. By hypothesis, there is an open neighborhood V of x such that $\overline{V} \subseteq U$. Then $U, X \setminus \overline{V}$ are disjoint open neighborhoods separating x and B respectively.

(2) The proof is analogous to (1).

It is quick to see that the subspace V of a Hausdorff space X is Hausdorff. Indeed, if $x, y \in V$, then there exist disjoint open neighborhoods $X \cap V, Y \cap V$ of x, y respectively, where X, Y are disjoint open neighborhoods of x, y in the space X.

It is also quick to see that the product X of Hausdorff spaces $\{X_\lambda\}_{\lambda \in \Lambda}$ is Hausdorff. Indeed, for distinct $\mathbf{x} = (x_\lambda)_{\lambda \in \Lambda}$ and $\mathbf{y} = (y_\lambda)_{\lambda \in \Lambda}$, there is some $j \in \Lambda$ for which $x_j \neq y_j$. Since X_j is Hausdorff, there exist disjoint, open neighborhoods $O_x, O_y \subseteq X_j$ of x_j, y_j. Then the sets

$$O_X = O_x \times \left(\prod_{\lambda \in \Lambda \setminus \{j\}} X_\lambda \right) \quad O_Y = O_y \times \left(\prod_{\lambda \in \Lambda \setminus \{j\}} X_\lambda \right),$$

are disjoint, open neighborhoods of **x**, **y** in the product topology.

[4.2.4] Theorem (*Subspace and Product of Regular Spaces is Regular*)

A subspace of regular spaces is regular. The product of a collection of regular spaces is regular.

Proof. Suppose X is a regular space and $Y \subseteq X$ is a subspace. Note Y is Hausdorff and thus T_1. Let x be a point in Y and B a closed subset of Y. Note that $\mathrm{Cl}_X(B) \cap Y = \mathrm{Cl}_Y(B) = B$, with the last equality by B closed. Considering $x \in X$ and $\overline{B} \subseteq X$ a closed set, by regularity of X we have that there exist disjoint, open neighborhoods U, V of x and $\overline{B}$. Then $U \cap Y, V \cap Y$ are disjoint, open neighborhoods of x and B, proving Y is regular.

Now suppose $\{X_\lambda\}_{\lambda \in \Lambda}$ is a collection of regular spaces. We show $X = \prod_{\lambda \in \Lambda} X_\lambda$ is regular by the alternative characterization. Note that X is Hausdorff and thus T_1, so it satisfies that part. Let $\mathbf{x} = (x_\lambda)_{\lambda \in \Lambda} \in X$, and let U be any open neighborhood of **x**. Since U is open, there is a basic, open set $\prod_\lambda U_\lambda$ such that $x \in \prod_\lambda U_\lambda \subseteq U$. From this basis, we can easily construct an open neighborhood V of x such that $\overline{V} \subseteq U$. We will define $V = \prod_{\lambda \in \Lambda} V_\lambda$. Indeed, for each $\lambda \in \Lambda$, consider U_λ. If $U_\lambda = X_\lambda$, then let $V_\lambda = X_\lambda$. Otherwise, let V_λ be some open set such that $\overline{V_\lambda} \subseteq U_\lambda$. Then V is an open neighborhood for x, and since $\overline{V} = \prod_{\lambda \in \Lambda} \overline{V_\lambda}$, we have that $\overline{V} \subseteq \prod_{\lambda \in \Lambda} U_\lambda \subseteq U$, such that X is regular. ❦

[4.2.5] Example (*Hausdorff But Not Regular*)

Consider $\mathbb{R}_K$, the reals endowed with the topology given by basis elements of the form $(a, b) \setminus K$, where $K = \{1/n\}_{n \in \mathbb{Z}_+}$. It is quick to see $\mathbb{R}_K$ is Hausdorff: for any $x \neq y \in \mathbb{R}_K$, let O_x, O_y be disjoint, open neighborhoods of x, y in the standard topology for $\mathbb{R}$, then note $O_x \setminus K, O_y \setminus K$ are disjoint, open neighborhoods of x, y in the K-topology. However, it is not regular by a straightforward proof.

Note that normality of a space is not preserved under subspace nor product. For products, consider the Sorgenfrey line $\mathbb{R}_\ell$, which is normal, and the Sorgenfrey plane $\mathbb{R}_\ell^2$, which is not normal.

[4.2.6] Recap

(1) Two points x, y are **topologically indistinguishable** if the collection of open neighborhoods about x is the exact same as the collection of open neighborhoods about y (otherwise, the points are **topologically distinguishable**). We say two points x, y are **separated** if there exist disjoint, open neighborhoods of x and y. The same idea applies to separating a point from a subset or a subset from a subset.

(2) Suppose X is a T_1 space. Then X is **regular** if any point x disjoint from a closed set B can be separated. X is **normal** if any pair of disjoint, closed sets A, B can be separated.

(3) The requirement that the sets are closed are important. If we drop that restriction, then there is limit point disjoint from a non-closed set, but separating them is impossible, so regularity would become far too restrictive. The same idea applies for normal spaces.

(4) We may also characterize regularity and normality in a different way. A space X is regular if and only if, for any open neighborhood U of x, there is some open neighborhood V of x such that $\overline{V} \subseteq X$. Analogously, a space X is normal if and only if, for any open neighborhood U of a closed set A, there is an open neighborhood V of A such that $\overline{V} \subseteq A$.

(5) Note that the equivalent definition of regularity closely resembles local compactness. Recall that a space X is **locally compact Hausdorff** if, for any open neighborhood U of x, there is some open neighborhood V of x such that $\overline{V}$ is compact and $\overline{V} \subseteq X$. At once, we have that locally compact Hausdorff spaces are regular.

(6) Subspace and product operations on regular spaces produce regular spaces. This is not true for normal spaces.

4.2.1 Exercises

[4.2.7] Problem

Show that if X is regular, then prove that any two distinct points in X have open neighborhoods whose closures are disjoint.

Proof. Let $x \neq y \in X$. By Hausdorffness of X, we have disjoint, open neighborhoods U_x, U_y about x, y respectively. By regularity of X, there are neighborhoods V_x, V_y of x, y such that $\overline{V_x} \subseteq U_x$ and $\overline{V_y} \subseteq U_y$, and so $\overline{V_x} \cap \overline{V_y} = \emptyset$, completing the proof. ❦

[4.2.8] Problem

Let $f, g : X \to Y$ be continuous maps into the Hausdorff space Y. Show the equalizer $\mathrm{Eq}(f, g) = \{x : f(x) = g(x)\}$ is closed in X.

Proof. We show $C = \mathrm{Eq}(f, g)$ contains all its limit points. For the sake of contradiction, suppose $x \in \overline{C} \setminus C$. By Hausdorffness of Y, there exist disjoint, open neighborhoods Y_f, Y_g of $f(x), g(x)$ respectively. Again by continuity of f, g, there exist open neighborhoods X_f, X_g of x for which $f(X_f) \subseteq Y_f$ and $g(X_g) \subseteq Y_g$. Then consider the single open neighborhood $U = X_f \cap X_g$ of x, which also satisfies $f(U) \subseteq Y_f$ and $g(U) \subseteq Y_g$. Since $x \in \overline{C}$, U intersects C at a point whose image under both f and g is y (since the point lies in the equalizer). Thus $y \in Y_f \cap Y_g$, a contradiction, and so the construction of x is invalid. Thus C contains all its limit points, completing the proof. ❦

4.3 Normal Spaces

Normality isn't a *super* nice condition (recall that subspaces and products of normal spaces are not necessarily normal). However, many familiar classes of topological spaces are normal, and normality turns out to be an important condition for certain big theorems like Urysohn's lemma and Tietze's extension theorem.

[4.3.1] THEOREM (*Regular, Second-Countable Spaces are Normal*)

A regular, second-countable space is normal.

Proof. Suppose X is a regular, second-countable space. Then X has a countable basis $\mathcal{B}$. Let A, B be any two closed subsets of X. Since A is closed, for any any $x \in A$, we may choose an open neighborhood U of x that is disjoint from B. By regularity, we know there is a neighborhood V of x for which $\overline{V} \subseteq U$. Since V is open, there is a basis element $O \in \mathcal{B}$ such that $x \in O \subseteq V$. Obviously, $\overline{O} \cap B = \emptyset$, since $\overline{O} \subseteq \overline{V}$. Identify each point x with some basis element as such, then let $\{U_n\}$ be the collection of all of these basis elements (note this is a subset of the countable basis, and so it is countable). Applying the same procedure to B, let the collection be $\{V_n\}$.

Now note $\bigcup U_n, \bigcup V_n$ are open sets containing A, B, but they are not necessarily disjoint. Instead, let us construct two open covers that *are* disjoint. For each n, define

$$U'_n = U_n \setminus \bigcup_{i=1}^{n} \overline{V_i} \quad V'_n = V_n \setminus \bigcup_{i=1}^{n} \overline{U_i}.$$

Note the finite union of compact spaces is compact, and as subspaces of a Hausdorff space, it is closed. Note each U'_n, V'_n is a complement of a closed set with respect to the open sets U_n, V_n, so they are open. Moreover, for any $x \in A$, there is some $n \in \mathbb{Z}_+$ where $x \in U_n$ and $x \notin \overline{V_i}$ for each $i \in \{1, \ldots, n\}$ (obviously, by $\overline{V_i}$ being disjoint from A), and vice-versa holds for B.

We have shown $U' = \bigcup U'_n$ and $V' = \bigcup V'_n$ are open sets containing A and B, and now we show they are disjoint. For the sake of contradiction, suppose $x \in U' \cap V'$, and so $x \in U'_j \cap V'_k$ for some $j, k \in \mathbb{Z}_+$. If $j \leq k$, then by definition of U'_j, we have that $x \in U_j$; by definition of V'_k, we have that $x \notin \overline{U_j}$, a contradiction. ❦

[4.3.2] THEOREM (*Metrizable Spaces Are Normal*)

A metrizable space is normal.

Proof. Let (X, d) be a metric space, and let $A, B \subseteq X$ be disjoint, closed sets. For each $a \in A$, there is some $\varepsilon_a > 0$ such that $B(a, \varepsilon_a) \cap B = \emptyset$, since otherwise a would be a limit point of B, forcing $a \in B$ by B being closed. An analogous procedure can be used to find ε_b for each $b \in B$. Finally, define

$$U = \bigcup_{a \in A} B(a, \varepsilon_a/2) \quad \text{and} \quad V = \bigcup_{b \in B} B(b, \varepsilon_b/2),$$

with $\varepsilon_a, \varepsilon_b$ as chosen before. Obviously, U and V form open sets that contain A and B. We prove they are disjoint by contradiction. Suppose $z \in U \cap V$, and so there exist $a \in A$ and $b \in B$ such that $z \in B(a, \varepsilon_a/2) \cap B(b, \varepsilon_b/2)$. Note that

$$d(a, b) \leq d(a, z) + d(z, b) < (\varepsilon_a + \varepsilon_b)/2.$$

If $\varepsilon_a \leq \varepsilon_b$, then $d(a, b) < \varepsilon_b$, and so $a \in B(b, \varepsilon_b/2)$. Vice-versa is true if $\varepsilon_b \geq \varepsilon_b$. But note the balls about points in B are disjoint from A by construction, and vice-versa for the balls constructed about points in A. Thus we have reached a contradiction, and so $U \cap V = \emptyset$, completing the proof. ❦

[4.3.3] Theorem (*Compact Hausdorff Spaces are Normal*)

A compact Hausdorff space is normal.

Proof. Suppose X is compact Hausdorff. We first show X is regular. Let x be any point in X and B be any disjoint, closed set. For any $b \in B$, we have the existence of open neighborhoods U_b and V_b of x and b that are disjoint. Note that $\{V_b\}$ is an open cover for B from open sets of X, and since B is a closed subset of a compact space, there is a finite subcover $\{V_k\}_{k=1}^n$. Then $U = \bigcup_{b \in B} U_b$ and $V = \bigcap_{k=1}^n V_k$ are disjoint, open sets containing x and B respectively. (Alternatively, note X is compact Hausdorff and thus locally compact Hausdorff, then apply a previous theorem.)

For any two disjoint, closed sets A, B, produce disjoint, open neighborhoods for each $a \in A$ and B. Intersect a finite subcover of the open neighborhoods covering B, then note it is an open neighborhood disjoint from the open neighborhood about A, which is the union of each open neigbhorhood about every $a \in A$. ❦

[4.3.4] Theorem

All well-ordered sets (totally ordered sets with the least upper bound property) are normal in their order topology.

4.4 The Urysohn Lemma

The Ursyohn lemma states that for any normal space X and any pair of closed subsets $A, B \subseteq X$, there exists a continuous $f : X \to [a, b]$ (for some $a, b \in \mathbb{R}$) such that $f|_A \equiv a$ and $f|_B \equiv b$.

[4.4.1] Theorem (*The Urysohn Lemma*)

Let X be a normal space, and let A, B be disjoint, closed subsets of X. Then there is a continuous map

$$f : X \to [0,1]$$

for which $f|_A \equiv 0$ and $f|_B \equiv 1$.

Before we prove this lemma, we consider a class of numbers known as **dyadic rationals**, a set of fractions whose denominator is a power of two. Namely, we consider the dyadic rationals contained in $[0,1]$. Let D_n be the set of dyadic rationals of order n contained in $[0,1]$ (i.e., pure fractions whose denominator is of the form 2^n). For example,

$$D_0 = \{0,1\}, \quad D_1 = \left\{0, \frac{1}{2}, 1\right\}, \quad D_2 = \left\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\right\}, \quad \cdots.$$

Suppose we bisect the interval $[0,1]$ n times. Then the endpoints of bisection correspond exactly with the values in D_n, motivating them geometrically. If we bisect the interval a countably infinite number of times (like we do in the proof of, say, Bolzano-Weierstrass), the endpoints yielded correspond exactly with the values in $D = \bigcup_{n \in \mathbb{Z}_+} D_n$. Crucially, D is dense in $[0,1]$, so the open sets indexed by D are fine enough to bring about a function that satisfies the hypotheses.

We will bisect $[0,1]$ a countably infinite number of times, use the shrinking lemma to force an open set each time we bisect, and index these open sets with the dyadic rationals. Then we will define a map using these open sets that satisfy the conditions of the hypothesis, completing the proof.

Proof. We start by creating our family of open sets indexed by D. We start with D_0: consider the set $U_1 = X \setminus B$. Then, since $A \cap B = 0$, we have that $A \subseteq U_1$. We have that A is closed and U_1 is open, and so by the shrinking lemma, there is an open set U_0 such that

$$A \subseteq U_0 \subseteq \overline{U_0} \subseteq U_1.$$

Now we move to D_1. Note that $\overline{U_0}$ is a closed subset of the open set U_1, so we apply the shrinking lemma again, yielding a set $U_{1/2}$ such that

$$\overline{U_0} \subseteq U_{1/2} \subseteq \overline{U_{1/2}} \subseteq U_1.$$

Now we move to D_2. Note that $\overline{U_0}$ is a closed subset of the open set $U_{1/2}$, and that $\overline{U_{1/2}}$ is a closed subset of the open set U_1. Then we have open sets $U_{1/4}, U_{3/4}$ such that

$$U_0 \subseteq \overline{U_0} \subseteq U_{1/4} \subseteq \overline{U_{1/4}} \subseteq U_{1/2} \subseteq \overline{U_{1/2}} \subseteq U_{3/4} \subseteq \overline{U_{3/4}} \subseteq U_1.$$

Repeat this process inductively, yielding the family of open sets $\{U_r\}_{r \in D}$. We are now ready to define the continuous function.

Define $f : X \to [0,1]$ by the function

$$f(x) = \inf\{r \in D : x \in U_r\} \quad (\inf(\emptyset) = 1).$$

For any $x \in X$, we identify the smallest dyadic rational r for which the open set U_r contains x. If no such U_r exists, then f emits 1. Immediately, we note that $f|_A \equiv 0$, since $A \subseteq U_0$ and so every element of A is contained in U_0. Conversely, we note that $f|_B \equiv 1$, since $U_1 \subseteq B$ and so no element of B is contained in $\{U_r\}$.

Finally, we prove that f is continuous, completing the proof. We show continuity on the subbasis elements $[0,a)$ and $(a,1]$ for any $a \in (0,1)$. Note that if $f(x) < a$ if and only if $x \in U_r$ for dyadic rationals r such that $r < a$. Thus

$$f^{-1}([0,a)) = \bigcup_{r \in D, r < a} U_r.$$

Conversely, suppose $f(x) > a$. By density of D in $[0,1]$, we can pick two dyadic rationals r, s such that $a < r < s < f(x)$. Since $s < f(x)$, we note that $x \notin U_s$. Since $\overline{U_r} \subseteq U_s$, note that $x \notin \overline{U_r}$ as well. Thus

$$f^{-1}((a,1]) = \bigcup_{r \in D, r > a} (X \setminus \overline{U_r}).$$

Thus the preimage of all subbasis elements are open, meaning f is continuous. We have completed the proof. ❦

[4.4.2] Remark (*Urysohn's Lemma Applies to Any Closed Interval*)

Note the argument involved is invariant under translation and scaling of the codomain. Thus Urysohn's lemma guarantees any continuous map $f : X \to [a, b]$ for which $f|_A \equiv a$ and $f|_B \equiv b$.

[4.4.3] Definition (*Separated by a Continuous Function*)

Suppose X is a topological space and $A, B \subseteq X$ are subsets. If there exists a continuous function $f : X \to [0, 1]$ such that $f|_A \equiv 0$ and $f|_B \equiv 1$, then we say that A and B can be **separated by a continuous function**.

[4.4.4] Theorem (*Normal iff. Disjoint, Closed Sets Separable by Continuous Function*)

A space X is normal if and only if, any pair of disjoint, closed sets can be separated by a continuous function.

Proof. ($\Longrightarrow$): This is the statement of Urysohn's Lemma.

($\Longleftarrow$): Suppose any disjoint, closed subsets $A, B \subseteq X$ can be separated by a continuous function f. Then note $f^{-1}([0, 1/2])$ and $f^{-1}((-1/2, 1])$ are disjoint open neighborhoods for A and B. ❦

Note that the shrinking lemma applies to normal spaces, and so the proof for Urysohn's lemma cannot be lifted to regular spaces. That is, the statement that a point and a disjoint, closed set can be separated by a continuous function is *false*. Instead, we define a subclass of regular spaces for which this is true.

[4.4.5] Definition (*Completely Regular*)

A space X that satisfies the T_1 condition is called **completely regular** if any singleton subset $\{x_0\}$ and some disjoint, closed set B can be separated by a continuous function.

Immediately, we note that a completely regular space is regular: if a singleton subset is separated from a disjoint, closed set by a continuous function f, then $f^{-1}([0, 1/2))$ and $f^{-1}((1/2, 1])$ are the desired disjoint, open neighborhoods about the singleton and closed set. Furthermore, we note that a normal space is completely regular, as we simply take one closed set to be a singleton.

This class of spaces is particularly nice as well: the subspace and product operations leave its property intact.

[4.4.6] Theorem (*Subspace and Product of Completely Regular Space is Completely Regular*)

Suppose X is a completely regular space. Then any subspace $U \subseteq X$ is also completely regular. Moreover, suppose $\{X_\lambda\}_{\lambda \in \Lambda}$ is a collection of completely regular spaces. Then the product $X = \prod_{\lambda \in \Lambda} X_\lambda$ is also completely regular.

Proof. Note that U is regular—subspaces of regular spaces are regular—and thus T_1. Let $x_0 \in U$ be any point that is disjoint from the closed set $B \subseteq U$. Since B is closed in U, it contains all its limit points in U and so x_0 is not a limit point for B, meaning $x_0 \notin \mathrm{Cl}_X(B)$. Thus x_0 is disjoint from the closed set $\mathrm{Cl}_X(B)$, where both objects are treated as elements of X. By complete regularity, there is a separation by a continuous map $f : X \to [0, 1]$. Then $g = f|_U$ is a continuous map that separates x_0 and $\mathrm{Cl}_X(B) \cap U = \mathrm{Cl}_U(B) = B$, making U completely regular.

The proof for the product being completely regular, like the subspace, is extremely similar to the proof of products being regular, so we omit the proof. ❦

[4.4.7] Remark (*The Separation Axioms*)

The separation axioms have alternative names of the form T_j. They are organized such that a T_j space is automatically T_i, and that there exists one T_i space that isn't T_j (for $j > i$). They are as follows.

$\mathbf{T}_0$ (*Kolmogorov*). Any two points are topologically distinguishable.

$\mathbf{T}_1$ (*Accessible*). Any two points have open neighborhoods that exclude the other point.

$\mathbf{T}_2$ (*Hausdorff*). Any two points can be separated.

$\mathbf{T}_3$ (*Regular*). A T_1 space for which any point and disjoint, closed set can be separated.

$\mathbf{T}_{3.5}$ (*Completely Regular*). A T_1 space for which any point and disjoint, closed set can be separated by a continuous function.

$\mathbf{T}_4$ (*Normal*). A T_1 space for which any pair of disjoint, closed sets can be separated.

$\mathbf{T}_5$ (*Completely Normal*). A T_1 space that is normal and has the property that all its subspaces are normal.

4.5 Urysohn's Metrization Theorem

We apply Urysohn's lemma to prove a foundational metrization theorem, called Urysohn's metrization theorem. This theorem posits that regular, second-countable spaces are metrizable by embedding them into a metric space (namely, $[0,1]^\omega \subseteq \mathbb{R}^\omega$). There are two variations of this proof, one where $\mathbb{R}^\omega$ is endowed with the product topology, the other where it is endowed with the uniform topology: we take up the proof using the product topology.

[4.5.1] Theorem (*Urysohn's Metrization Theorem*)

Suppose X is a regular, second-countable space. Then X is metrizable.

Proof. We show X is metrizable by embedding it into a metrizable space Y. If f is the appropriate embedding, note that $(X \cong f(X)) \subseteq Y$. Since $f(X)$ is a subspace of the metric space Y, $f(X)$ takes on a metric topology, and by homeomorphism so does X. As aforementioned, we let Y be the space $\mathbb{R}^\omega$ together with its product topology, and we show there is an embedding of X into $[0,1]^\omega$.

We first prove that there exists a countable collection of continuous functions $f_n : X \to [0,1]$ such that, for any point $x_0 \in X$ and any neighborhood U of x_0, there exists some $n \in \mathbb{Z}_+$ such that f_n is positive at x_0 and vanishes outside of U. Since X is regular and second-countable, it is normal. By Urysohn's lemma, for any fixed $x_0 \in X$ and neighborhood U of x_0, the disjoint closed sets $\{x_0\}$ and $X \setminus U$ can be separated by a continuous function $f : X \to [0,1]$ such that $f(x_0) = 1$ and $f(X \setminus U) = \{0\}$. We now go one step further to show that we can extract a single *countable* collection of such functions that works for every possible (x_0, U) pair simultaneously.

Let $\{B_n\}_{n \in \mathbb{Z}_+}$ be a countable basis for X. For each pair of indices $m, n \in \mathbb{Z}_+$ for which $\overline{B}_n \subseteq B_m$, we apply Urysohn's lemma to separate the disjoint closed sets $\overline{B}_n$ and $X \setminus B_m$ by a continuous map $g_{m,n} : X \to [0,1]$ satisfying $g_{m,n}(\overline{B}_n) = \{1\}$ and $g_{m,n}(X \setminus B_m) = \{0\}$. Because this collection $\{g_{m,n}\}$ is indexed by a subset of $\mathbb{Z}_+ \times \mathbb{Z}_+$, it is countable and can be reindexed as $\{f_n\}_{n \in \mathbb{Z}_+}$.

To see that this collection satisfies the requirement, let $x_0 \in X$ and let U be an open neighborhood of x_0. By the definition of a basis, there is some B_m such that $x_0 \in B_m \subseteq U$. By regularity of X, there is some basis element B_n such that $x_0 \in B_n \subseteq \overline{B}_n \subseteq B_m$. The function $g_{m,n}$ is therefore defined for this pair; it yields $g_{m,n}(x_0) = 1 > 0$ and vanishes on $X \setminus B_m \supseteq X \setminus U$, completing the proof.

Now consider the map $F : X \to \mathbb{R}^\omega$ defined by $F(x) = (f_k(x))_{k \in \mathbb{Z}_+}$. At once we note that F is continuous since each component is continuous, a nice property of the product topology. Moreover, it is injective: for any $x \neq y$ in X, there exists an open neighborhood U of x that excludes y (by X being T_1). Then there is some $n \in \mathbb{Z}_+$ such that $f_n(x) > 0$ and $f_n(X \setminus U) = 0$ (and so $f(y) = 0$), meaning $F(x) \neq F(y)$. Thus F is a continuous bijection onto its image $Z = F(X)$, and our final task is to show F maps open sets to open sets.

Suppose $O \subseteq X$ is open: we seek to show $F(O)$ is open in $\mathbb{R}^\omega$. Since F is a bijection, there is some unique $x_0 \in X$ such that $F(x_0) = z_0$. Let $N \in \mathbb{Z}_+$ be the index for which $f_N(x_0) > 0$ and $f_n(X \setminus O) = \{0\}$. Finally, let

$$W = \underbrace{(\pi_N^{-1}((0,\infty)))}_{:=V} \cap Z.$$

Note that $(0,\infty)$ is open in $\mathbb{R}$, and by continuity of the projection map π_N, we have that V is an open subset of $\mathbb{R}^\omega$. Thus W is an open subset of Z in its subspace topology. We show that

$$z_0 \in W \subseteq F(O),$$

implying that $F(O)$ is open. Since $f_N(x_0) > 0$, we have that $z_0 \in [\pi_N^{-1}((0,\infty)) = V]$, and $z_0 \in Z$ is immediate by F being a bijection—thus $z_0 \in W$. Now let $w \in W$ be arbitrary. Since $Z \subseteq W$, there is some $x \in X$ such that $F(x) = z$, and so $\pi_N(z) \in (0,\infty)$. Then

$$\pi_N(z) = \pi_N(F(x)) = f_N(x) \in (0,\infty),$$

meaning $f_N(x) > 0$ and so $x \in O$. Thus $w \in F(O)$, meaning $W \subseteq F(O)$, and so $F(O)$ is open. Thus F is a homeomorphism onto an image (an embedding), completing the proof that X is metrizable. ❦

[4.5.2] Theorem (*Embedding Theorem*)

Let X be a T_1 space. Suppose that $\{f_\lambda\}_{\lambda\in\Lambda}$ is a collection of continuous maps $f: X \to \mathbb{R}$ satisfying the condition that, for any $x_0 \in X$ together with an open neighborhood U of x_0, there is some index $\lambda \in \Lambda$ such that $f_\lambda(x_0) > 0$ and $f_\lambda(X \setminus U) = \{0\}$. Then the function $F: X \to \mathbb{R}^\Lambda$, with codomain taking the product topology, defined by

$$F(x) = (f_\lambda(x))_{\lambda\in\Lambda}$$

is an embedding of X into $\mathbb{R}^\Lambda$.

The proof is drawn directly from the second part of the previous proof, simply replacing the indexing set $\mathbb{Z}_+$ with Λ. If each f_λ maps into $[0,1]$, as the case with the last proof, then we note F is an embedding of X into $[0,1]^\Lambda$, called the **Hilbert cube** in $\mathbb{R}^\Lambda$.

4.6 Tietze Extension Theorem

Another useful consequence of the Urysohn lemma is the useful theorem called the Tietze extension theorem. A common problem in analysis is being able to extend a construction to a larger space, while still keeping important properties intact. For a space X, we may have a continuous, real-valued function $f: A \to \mathbb{R}$, with A a subset of X. We ask if there is a way to extend f to some function $\tilde{f}: X \to \mathbb{R}$, such that $\tilde{f}|_A = f$ and that continuity is preserved. Of course, this is not true for all spaces X, but certain restrictions on X and A make this theorem true. This is what the Tietze extension theorem is concerned with.

[4.6.1] Theorem (*Tietze Extension Theorem*)

Suppose X is a normal space and $A \subseteq X$ is closed. Then any continuous map $f: A \to [a,b]$ may be extended to a continuous map $f: X \to [a,b]$. The result holds if the closed interval in the codomain is replaced with $\mathbb{R}$.

The proof will be omitted for now.

4.7 Embeddings of Manifolds

We have shown that regular, second-countable spaces can be embedded into $\mathbb{R}^\omega$. A natural question is to ask which type of space X can be embedded into some finite-dimensional Euclidean space $\mathbb{R}^n$. We show that compact manifolds satisfy this property.

[4.7.1] Definition (*m-Manifold*)

A space X is said to be an m**-manifold** if X is Hausdorff, second-countable, and is locally homeomorphic to $\mathbb{R}^m$. That is, for any $x \in X$, there is some open neighborhood U of x such that U is homeomorphic to some open subset of $\mathbb{R}^m$.

[4.7.2] Example (*Examples of Manifolds*)

A curve (with some regularity conditions imposed) is a 1-manifold, and the same for a surface being a 2-manifold.

[4.7.3] Definition (*Support of a Map*)

Suppose $\varphi : X \to \mathbb{R}$ is a real-valued map. Then the **support** of φ, denoted by $\operatorname{supp}(\varphi)$, is the closure of $\varphi^{-1}(\mathbb{R} \setminus \{0\})$.

[4.7.4] Remark (*Intuition behind Support*)

The support of a function is not simply where the points where the map is nonzero: it's the points where there are no open neighborhoods for which the map vanishes on. We can characterize the support of a function as where it's "active". A polynomial p may be zero at a root $x = r$, but it either rebounds or intersects without "wasting any time," and so we would say that the polynomial is "active" at r, and so $r \in \operatorname{supp}(p)$. The bottom line is, if a point x is not in $\operatorname{supp}(f)$, then there is some open neighborhood U of x such that $f|_U \equiv 0$. Thus the closure makes the support a much better tool to determine where a function is dead. We also note that the support being closed is a useful property (analogously, the complement of the support being open is useful).

[4.7.5] Definition (*Partition of Unity*)

Let $\{U_1, \ldots, U_n\}$ be some finite covering of the space X. Then a collection of continuous maps $\{\varphi_1, \ldots, \varphi_n\}$ such that $\varphi_k : X \to [0,1]$ is said to be a **partition of unity** dominated by $\{U_k\}$ if it satisfies the following conditions.

(1) $\operatorname{supp}(\varphi_k) \subseteq U_k$ for each $k \in \{1, \ldots, n\}$.

(2) For any $x \in X$, $\sum_{k=1}^{n} \varphi_k(x) = 1$.

[4.7.6] Remark (*Intuition behind Partition of Unity*)

Suppose X has a finite cover $\{U_1, \ldots, U_n\}$ together with a collection of continuous local maps $\{g_1, \ldots, g_n\}$ where each $g_k : U_k \to \mathbb{R}$. A natural question is: how can we produce a global map $g : X \to \mathbb{R}$ that represents these local maps best? There is no guarantee that g_j and g_k agree on the overlap $U_j \cap U_k$, so using the pasting lemma is not an option.

A naive approach would be to use characteristic functions χ_{U_k} to toggle the maps on and off, defining $g(x) = \sum_{k=1}^{n} \chi_{U_k}(x) g_k(x)$ (where the product is taken to be 0 outside U_k). However, this fails for two reasons. First, the binary nature of the characteristic function causes discontinuities at the boundaries. If a sequence of points inside U_k approaches a boundary point $p \in \partial U_k$, the term $\chi_{U_k}(x) g_k(x)$ approaches $g_k(p)$, which is generally nonzero. But for points immediately outside the boundary, the term is exactly 0. Unless the local map g_k happens to naturally vanish at the boundary of U_k, this creates a jump discontinuity. Second, in overlapping regions where multiple sets intersect, the values are blindly added together. If $x \in U_j \cap U_k$, the sum yields $g_j(x) + g_k(x)$, double-counting the outputs and artificially inflating the result.

A partition of unity solves this by acting as a continuous approximation of these characteristic functions. Rather than assigning a rigid binary 1 or 0 to dictate which open set x belongs to, $\varphi_k(x)$ assigns a “fractional ownership” to U_k. Deep inside U_k, away from other sets in the cover, $\varphi_k(x) = 1$ and acts exactly like a standard indicator function. In an overlap $U_j \cap U_k$, however, the functions provide a continuous interpolation that smoothly crossfades between the patches. For instance, suppose an overlap $V = U_j \cap U_k$ intersects no other sets U_i; by the first condition, all other maps φ_i must vanish identically on V. For any $v \in V$, the value $\varphi_j(v)$ measures how much “control” U_j exerts over v, and similarly for $\varphi_k(v)$. The second condition then forces $\varphi_j(v) + \varphi_k(v) = 1$, ensuring that the total ownership of the point is exactly 100%. Simply put, each map φ_k determines the amount of control U_k exerts over any point $x \in X$, with the restriction that U_k claims no ownership outside its boundary and that the the ownership each map exerts over x always sums perfectly to 1.

We define the global map as $g(x) = \sum_{k=1}^{n} \varphi_k(x) g_k(x)$, a convex combination of the local maps with the maps from the partition of unity. Since $\sum \varphi_k(x) = 1$, this behaves as a true weighted average. If the local maps g_j and g_k both output a value of c at an overlap, the global map outputs $0.7c + 0.3c = c$, which successfully blends conflicting local maps without inflating or deflating the values.

We will eventually prove compact manifolds may be embedded into Euclidean space by taking a finite cover of open sets, constructing local embedding maps from each open set to Euclidean space, then using a partition of unity to stitch together these maps and create an embedding of the manifold into Euclidean space.

[4.7.7] Theorem (*Existence of Partition of Unity*)

Let $\{U_1, \dots, U_n\}$ be a finite open covering for the normal space X. Then X admits a partition of unity dominated by $\{U_k\}_{k=1}^n$.

Proof. We first show that any finite cover $\{U_k\}_{k=1}^n$ of X admits a finite cover $\{V_k\}_{k=1}^n$ such that $\overline{V_k} \subseteq U_k$ for each $k \in \{1, \dots, n\}$. Define

$$A = U \setminus (U_2 \cup \cdots U_n).$$

Note that A is a closed subset of U_1, and by normality, there exists an open set V_1 such that $A \subseteq V_1 \subseteq \overline{V_1} \subseteq A$. Thus $\{V_1, U_2, \dots, U_n\}$ is a finite open cover for U, since the missing elements from the incomplete cover $\{U_2, \dots, U_n\}$ are in A, which is a subset of V_1. In general, if

$$\{V_1, \dots, V_{k-1}, U_k, U_{k+1}, \dots, U_n\}$$

covers X, let

$$A = X \setminus (V_1 \cup \cdots \cup V_{k-1} \cup U_{k+1} \cup \cdots \cup U_n).$$

Apply the same procedure to find V_k, then note $\{V_1, \dots, V_k, U_{k+1}, \cdots, U_n\}$ covers X. Induction up to the n-th step shows this result is true.

Finally, let X be a normal space together with a finite open cover $\{U_1, \dots, U_n\}$. By the previous theorem, choose $\{V_k\}_{k=1}^m$ and $\{W_k\}_{k=1}^m$ such that $\overline{V_k} \subseteq U_k$ and $\overline{W_k} \subseteq V_k$ for each $k \in \{1, \dots, n\}$. Note that $\overline{W_k}$ and $X \setminus V_k$ are disjoint, closed sets in a normal space X, so Urysohn's lemma guarantees a continuous map $\psi_k : X \to [0,1]$ such that $\psi_k(\overline{W_k}) = \{1\}$ and $\psi_k(X \setminus V_k) = \{0\}$. Thus $\psi_k^{-1}(\mathbb{R} \setminus \{0\}) \subseteq V_k$, and so

$$\operatorname{supp}(\psi_k) \subseteq \overline{V_k} \subseteq U_k.$$

Define $\psi(x) = \sum_{k=1}^n \psi_k(x)$, which is strictly positive (find k such that $x \in U_k$, then note $\psi_k(x) > 0$). Then define

$$\varphi_k(x) = \frac{\psi_k(x)}{\psi(x)},$$

and note $\{\varphi_k\}_{k=1}^n$ is a partition of unity for X.

[4.7.8] Theorem

A compact manifold can be embedded into finite-dimensional Euclidean space.

Proof. Let X be a compact m-manifold. Note that by compactness and by m-manifolds being locally homeomorphic to $\mathbb{R}^m$, there exists a finite open cover $\{U_k\}_{k=1}^n$ comprised of open sets that are homeomorphic to open subsets of $\mathbb{R}^m$. Thus, for each $k \in \{1, \dots, n\}$, there exists a map $g_k : U_k \to \mathbb{R}^m$ that is an embedding.

Since X is a compact manifold, it is compact Hausdorff and thus normal, so it admits a partition of unity $\{\varphi_k\}_{k=1}^n$ dominated by $\{U_k\}_{k=1}^n$. For each $k \in \{1, \dots, n\}$, define $h_k : X \to \mathbb{R}^m$ by

$$h_k(x) = \begin{cases} \varphi_k(x) \cdot g_k(x) & x \in U_k, \\ \mathbf{0}_m & x \in X \setminus \operatorname{supp}(\varphi_k). \end{cases}$$

Note that on the intersection $U_k \cap (X \setminus \operatorname{supp}(\varphi_k))$, we have $\varphi_k(x) = 0$, so both branches agree with $\mathbf{0}_m$. Because U_k and $X \setminus \operatorname{supp}(\varphi_k)$ are open sets whose union is X, each h_k is well-defined and continuous by the pasting lemma.

Finally, set $N = n + nm$ and define the global map

$$g : X \to \mathbb{R}^N \cong \left(\underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{n \text{ times}} \times \underbrace{\mathbb{R}^m \times \cdots \times \mathbb{R}^m}_{n \text{ times}} \right)$$

by

$$g(x) = (\varphi_1(x), \dots, \varphi_n(x), h_1(x), \dots, h_n(x)).$$

We show that g is an embedding of X into $\mathbb{R}^N$. At once we have that g is continuous, since each of its component functions φ_k and h_k is continuous.

Next, we show that g is injective. Suppose $x, y \in X$ such that $g(x) = g(y)$. This implies $\varphi_k(x) = \varphi_k(y)$ and $h_k(x) = h_k(y)$ for every

$k \in \{1, \ldots, n\}$. Because $\{\varphi_k\}$ is a partition of unity, we have $\sum_{k=1}^{n} \varphi_k(x) = 1$, so there must exist at least one index k for which $\varphi_k(x) > 0$. Since $\varphi_k(y) = \varphi_k(x) > 0$, both points x and y lie in $\operatorname{supp}(\varphi_k) \subseteq U_k$. Evaluating h_k on U_k yields

$$\varphi_k(x) \cdot g_k(x) = h_k(x) = h_k(y) = \varphi_k(y) \cdot g_k(y).$$

Since $\varphi_k(x) = \varphi_k(y) > 0$, we may divide by this nonzero scalar to obtain $g_k(x) = g_k(y)$. Because g_k is an embedding on U_k, it is injective, forcing $x = y$. Thus, g is an injective continuous map.

Lastly, because X is compact and $\mathbb{R}^N$ is Hausdorff, the continuous injective map $g : X \to g(X)$ is a closed map, and therefore a homeomorphism onto its image. This completes the proof that g is an embedding into $\mathbb{R}^N$. ❦

5 The Tychonoff Theorem

5.1 The Tychonoff Theorem

[5.1.1] THEOREM (*Tychonoff's Theorem*)

An arbitrary product of compact spaces is compact in the product topology.

6 Paracompactness

6.1 Local Finiteness

[6.1.1] Definition (*Local Finiteness*)

Let X be a topological space together with a collection of subsets $\mathcal{A}$. We say $\mathcal{A}$ is **locally finite** if every point $x \in X$ has some open neighborhood U of x such that only finitely many members of $\mathcal{A}$ intersect U.

[6.1.2] Example

Note that $\mathcal{A} = \{(n, n+1)\}_{n \in \mathbb{Z}_+}$ is a locally finite collection with respect to $\mathbb{R}$. Indeed, for any $x \in X$, the open neighborhood $(x - 0.5, x + 0.5)$ is intersected by, at most, three members of $\mathcal{A}$. Also note that $\{(0, 1/n)\}_{n \in \mathbb{Z}_+}$ is a locally finite collection on $(0,1)$, but not on $\mathbb{R}$ (since 0, 1 are limit points and thus admit countably many elements of this collection into any neighborhood of those points).

[6.1.3] Theorem (*Subset and Closure Properties of Locally Finite Collections*)

Let $\mathcal{A}$ be a locally finite collection of subsets of X.

(1) Any subcollection of $\mathcal{A}$ is locally finite.

(2) The collection $\mathcal{B} = \{\overline{A}\}_{A \in \mathcal{A}}$ is a locally finite collection.

(3) $\overline{\bigcup_{A \in \mathcal{A}} A} = \bigcup_{A \in \mathcal{A}} \overline{A}$.

Proof.

(1): Immediate.

(2): Let $x \in X$ be arbitrary. Choose an open neighborhood U of x such that the only members of $\mathcal{A}$ that intersect U are $\{A_k\}_{k=1}^n$. Let $m \notin \{1, \ldots, n\}$ such that $A_m \in \mathcal{A}$, then note $U \cap \overline{A_m} = \emptyset$ since $U \cap A_m = \emptyset$. Thus U intersects finitely many elements of $\mathcal{B}$.

(3): Let $Y = \bigcup_{A \in \mathcal{A}} A$. It is a standard topological fact that $\bigcup_{A \in \mathcal{A}} \overline{A} \subseteq \overline{Y}$. We prove the reverse inclusion. Suppose $x \in \overline{Y}$. By local finiteness, there is some open neighborhood U of x that intersects only finitely many members of $\mathcal{A}$, say $A_1, \ldots, A_k$. For the sake of contradiction, suppose $x \notin \bigcup_{A \in \mathcal{A}} \overline{A}$. Then x is not in any of $\overline{A_1}, \ldots, \overline{A_k}$. Thus $V = U \setminus (\overline{A_1} \cup \cdots \cup \overline{A_k})$ is an open neighborhood of x. But V intersects no elements of $\mathcal{A}$, meaning it does not intersect Y, which contradicts $x \in \overline{Y}$. Thus $x \in \bigcup_{A \in \mathcal{A}} \overline{A}$, completing the proof.

[6.1.4] Definition (*Refinement, Countably Locally Finite*)

Let $\mathcal{A}$ be a covering of X. A collection $\mathcal{B}$ is a **refinement** of $\mathcal{A}$ if for each $B \in \mathcal{B}$, there is some $A \in \mathcal{A}$ such that $B \subseteq A$. If the elements of $\mathcal{B}$ are open, we call it an open refinement.

Furthermore, a collection $\mathcal{B}$ is **countably locally finite** (or σ-locally finite) if it can be written as a countable union of locally finite collections.

6.2 Paracompact Spaces

[6.2.1] Definition (*Paracompactness*)

A space X is said to be **paracompact** if every open covering $\mathcal{A}$ of X admits a locally finite open refinement $\mathcal{B}$ that covers X.

[6.2.2] Remark (*Intuition behind Paracompactness*)

Paracompactness is a natural generalization of compactness. Instead of demanding a finite subcover (which is often too strong of a global condition), we demand a locally finite open refinement. This ensures that while the total number of open sets in the cover might be infinite, locally around any point, the space only "sees" finitely many sets.

As we will see, this is precisely the topological machinery required to generalize partitions of unity. Because the refinement is locally finite, the seemingly infinite sum $\sum \varphi_k(x)$ evaluates to a finite sum in some neighborhood of any point x. Thus, the sum is automatically well-defined and continuous for free, without any need for analytic convergence tests.

[6.2.3] Theorem (*Paracompact Hausdorff Spaces are Normal*)

Every paracompact Hausdorff space is normal.

[6.2.4] Theorem (*Stone's Theorem*)

Every metrizable space is paracompact.

We omit the proofs of these theorems as they are highly technical, but their consequences are immense. Specifically, they allow us to generalize the existence of partitions of unity from compact spaces to any paracompact space.

[6.2.5] Theorem (*Existence of Partitions of Unity*)

Let X be a paracompact Hausdorff space, and let $\{U_\alpha\}_{\alpha\in J}$ be an indexed open covering of X. Then there exists a partition of unity $\{\varphi_\alpha\}_{\alpha\in J}$ dominated by $\{U_\alpha\}$.

This theorem is fundamentally why paracompactness is the standard assumption in differential geometry and functional analysis. It guarantees we can take locally defined objects (like coordinate charts, metric tensors, or differential forms) and stitch them together into global objects seamlessly, even if the manifold is not compact.

6.3 Metrization Theorems

The Urysohn metrization theorem gave us a sufficient condition for metrizability (regular and second-countable), but mathematicians are rarely satisfied until conditions are both necessary and sufficient. The following theorems, whose proofs we omit as they are largely technical extensions of Urysohn's methods, provide exact characterizations of metrizable spaces.

[6.3.1] Theorem (*Nagata-Smirnov Metrization Theorem*)

A space X is metrizable if and only if X is regular and admits a countably locally finite basis.

[6.3.2] Definition (*Locally Metrizable*)

A space X is **locally metrizable** if every point $x \in X$ has an open neighborhood U that is metrizable in the subspace topology.

[6.3.3] Theorem (*Smirnov Metrization Theorem*)

A space X is metrizable if and only if it is a paracompact Hausdorff space that is locally metrizable.

[6.3.4] Remark (*Manifolds are Metrizable*)

Smirnov's theorem is particularly elegant when applied to manifolds. Because an m-manifold is locally homeomorphic to $\mathbb{R}^m$, it is automatically locally metrizable. Thus, a manifold is globally metrizable if and only if it is paracompact Hausdorff. Since standard definitions of manifolds require them to be second-countable (which implies they are Lindelöf, and regular Lindelöf spaces are paracompact), manifolds are automatically metrizable!

7 Complete Metric Spaces and Function Spaces

7.1 Complete Metric Spaces

Before we proceed, we will revisit standard metrics and properties about them.

[7.1.1] Recap

(1) $\mathbb{R}^n$ has two common metrics: the **Euclidean metric**

$$d(x,y) = \left[\sum_{i=1}^{n}(x_i - y_i)^2\right]^{1/2}$$

and the **square metric** $\rho(x,y) = \max_{1\le i\le n}|x_i - y_i|$.

(2) By simple geometry, note that $\rho(x,y) \le d(x,y) \le \sqrt{n}\rho(x,y)$ for $x,y \in \mathbb{R}^n$. Thus properties of ρ (such as sequence convergence, sequences being Cauchy, the topology generated by the metric) are brought to d immediately by this inequality. Note that the constant $\sqrt{n}$ is the only obstruction, and it is exactly what fails in infinite dimensions (since $\sqrt{n} \to \infty$ as $n \to \infty$).

(3) It is easy to show the metric topology generated by ρ coincides perfectly with the product topology on $\mathbb{R}^n$. By the inequality from before, the metric topology generated by d also coincides with the product topology on $\mathbb{R}^n$.

(4) Note that neither metric generalizes cleanly to infinite products, since both may take the value ∞. Instead, given *any* metric d on a space X, define the **standard bounded metric**

$$\overline{d}(x,y) = \min\{d(x,y), 1\}.$$

This induces the same topology as d, so boundedness is not a topological property; it depends on the choice of metric. Truncating in this way costs nothing topologically and is what makes the infinite constructions below well-defined.

(5) We now generalize the square metric to $\mathbb{R}^\omega$. The most direct approach is the **uniform metric**

$$\overline{\rho}(x,y) = \sup_{n\in\mathbb{Z}_+}\{\overline{d}(x_n,y_n)\},$$

replacing max with sup and truncating the distances above by 1. The topology this induces is the **uniform topology**. Alternatively, we may define the **product metric**

$$D(x,y) = \sup_{n\in\mathbb{Z}_+}\{\overline{d}(x_n,y_n)/n\},$$

which is the same as the uniform metric, but damps the nth coordinate so that later coordinates are less "important." Fittingly, it is the metric that produces the product topology on $\mathbb{R}^\omega$.

(6) These metrics can be placed on the space Y^X, the set of functions $f : X \to Y$. Then the product metric sets the mode of convergence to pointwise on this function space, whereas the uniform metric sets the mode to uniform convergence.

(7) In defining metrics on function spaces, consider the space $B(X,Y)$ of bounded functions $f : X \to Y$. We may define the **sup metric**

$$\rho(f,g) = \sup_{x\in X}\{d(f(x), g(x))\}.$$

Note that we do not need to use the bounded metric $\overline{d}$, since the functions are already bounded above, and so sup works out of the box. In general, note that the relationship between the uniform and sup metrics are simply $\overline{\rho}(f,g) = \min\{\rho(f,g), 1\}$.

(8) Note that the product topology is strictly coarser than the uniform topology, which is strictly coarser than the box topology.

[7.1.2] Definition (*Cauchy Sequence, Complete Metric Space*)

Let (X,d) be a metric space. A sequence $(x_n) \subseteq X$ is said to be **Cauchy** if, for any $\varepsilon > 0$, there is some $N \in \mathbb{Z}_+$ such that $d(x_m, x_n) < \varepsilon$ for every $m, n \geq N$. We say the metric space X is **complete** if every Cauchy sequence converges.

[7.1.3] Theorem (*Equivalence of Convergent Sequences and Cauchy Sequences in Metric Spaces*)

In a complete metric space (X,d), a sequence converges if and only if it is Cauchy.

This fact is important in analysis since, in complete metric spaces, we can demonstrate a sequence converges if it is Cauchy, without needing a candidate for the limit.

[7.1.4] Theorem (*Closed Subspace of Complete Metric Space is Complete*)

A closed subspace of a complete metric space is complete.

Proof. Of course, a subspace topology derived from the metric topology is metric. Consider any Cauchy sequence contained in the closed subspace. It necessarily converges to a limit point of the closed subspace, but closed spaces contain all their limit points, and so the Cauchy sequence converges to a point in the closed subspace. Thus the closed subspace is complete. ❦

[7.1.5] Theorem (*Complete if Every Cauchy Sequence has Accumulation Point*)

A metric space (X,d) is complete if every Cauchy sequence has a convergent subsequence.

Proof. Suppose if $(x_n) \subseteq X$ is a Cauchy sequence with a subsequence $x_{n_k} \to x$. For any $\varepsilon > 0$, note there is some $N_1 \in \mathbb{Z}_+$ such that $d(x_n, x_{n_k}) < \varepsilon/2$ for every $n, n_k \geq N_1$, since the sequence is Cauchy. Moreover, there is some $N_2 \in \mathbb{Z}_+$ such that $d(x_{n_k}, x) < \varepsilon/2$ for every $n_k \geq N_2$ by definition of convergence. Thus, for any $n \geq \max\{N_1, N_2\}$, we have that

$$d(x_n, x) \leq d(x_n, x_{n_k}) + d(x_{n_k}, x) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Since all Cauchy sequences are convergent, the space is complete. ❦

[7.1.6] Theorem (*Euclidean Space is Complete in its Standard Metrics*)

Euclidean space is complete under the standard and square metrics.

Proof. Let $(\mathbb{R}^n, \rho)$ be Euclidean space endowed with the square metric. For any Cauchy sequence x_n in this metric space, there is some $N \in \mathbb{Z}_+$ such that $\rho(x_m, x_n) < 1$ for every $m, n \geq N$. Then choose a number

$$M = \max\{\rho(x_0, \mathbf{0}_n), \rho(x_1, \mathbf{0}_n), \ldots, \rho(x_{N-1}, \mathbf{0}_n), \rho(x_N, \mathbf{0}_n) + 1\}.$$

Note then that x_n is fully contained in $[-M, M]^n$. Note this box is compact, and so x_n has a convergent subsequence by sequential compactness. Thus Cauchy sequences converge and so the space is complete.

Finally, for $(\mathbb{R}^n, d)$, note that a sequence converges if and only if it converges with respect to ρ, and the same for the sequence being Cauchy. ❦

[7.1.7] Theorem (*Sequence in Product Space Converges iff. Components Converge*)

Let $\{X_\lambda\}_{\lambda\in\Lambda}$ be an arbitrary collection of spaces with $X = \prod_{\lambda\in\Lambda} X_\lambda$ its product. Then a sequence $\mathbf{x}_n \to \mathbf{x}$ in the product space if and only if $\pi_\lambda(\mathbf{x}_n) \to \pi_\lambda(\mathbf{x})$ for every $\lambda \in \Lambda$.

Proof.

($\Longrightarrow$): Note that π_λ is continuous for every $\lambda \in \Lambda$, so the image of $\mathbf{x}_n$ under each projection maps to the projection of $\mathbf{x}$.

($\Longleftarrow$): Let $\mathbf{x}_n$ be some sequence such that there exists some $\mathbf{x} \in X$ for which $\pi_\lambda(\mathbf{x}_n) \to \pi_\lambda(\mathbf{x})$ for every $\lambda \in \Lambda$. Let U be any basic open neighborhood of $\mathbf{x}$. By definition, we have that there exist indices $\{\lambda_k\}_{k=1}^n$ such that

$$U = \bigcap_{k=1}^{n} \pi_\lambda^{-1}(U_\lambda),$$

where $U_{\lambda_k} \subseteq X_{\lambda_k}$ is open. For each $k \in \{1,\dots,n\}$, there is some $N_k \in \mathbb{Z}_+$ such that $\pi_{\lambda_k}(\mathbf{x}_n) \in \pi_{\lambda_k}(U_\lambda)$ for every $n \geq N_k$. Choosing $N = \max\{N_k\}_{k=1}^n$, note that $\mathbf{x}_n \in U$ for every $n \geq N$, completing the proof. ❦

[7.1.8] Theorem (*$(\mathbb{R}^\omega, D)$ is complete*)

The metric space $(\mathbb{R}^\omega, D)$, with $D(x,y) = \sup_{n\in\mathbb{Z}_+}\{\overline{d}(x,y)/n\}$ is complete.

Proof. Suppose $\mathbf{x}_n$ is Cauchy in this metric space. Note that

$$\overline{d}(\pi_n(\mathbf{x}_n), \pi_n(\mathbf{y}_n)) \leq nD(\mathbf{x},\mathbf{y}),$$

and so $\pi_n(\mathbf{x}_n)$ is a Cauchy sequence in $(\mathbb{R}, \overline{d})$ for each $n \in \mathbb{Z}_+$. This metric space is complete, and so the projections of the sequence converge. By the previous theorem, this means $\mathbf{x}_n$ converges. ❦

[7.1.9] Example (*Completeness is not a Topological Property*)

Note that $\mathbb{R}$ is complete in its usual metric. Now consider the subspace $(-1,1)$ with the same metric, $d(x,y) = |x-y|$. But note the Cauchy sequence $x_n = 1 - 1/n$ converges to 1, which is outside of this interval. Thus this space is not complete, even though $(-1,1) \cong \mathbb{R}$. Thus completeness is not a topological property.

[7.1.10] Definition (*Uniform Metric, Revisited*)

Let (Y,d) be a metric space, together with an induced bounded metric $\overline{d}(x,y) = \min\{d(x,y),1\}$. Let Y^X be the space of functions $\{f \mid f : X \to Y\}$. Then we define the **uniform metric**

$$\overline{\rho}(f,g) = \sup_{x\in X}\{d(f(x), g(x))\}.$$

[7.1.11] Remark

Note that if X is an indexing set, the space Y^X is the set of sequences in Y indexed by X. Then the uniform metric defined for sequences coincides with the uniform metric defined in terms of functions. Of course, this is all true because a sequence is truly a function from the indexing set into Y.

[7.1.12] Theorem (*Criterion for Complete Metric Space Under Unifom Metric*)

Suppose (Y,d) is a complete metric space. Then the metric space $(Y^X, \overline{\rho})$ is complete as well.

Proof. Let $(f_n)_{n\in\mathbb{Z}_+}$ be a Cauchy sequence in Y^X, where each $f_n : X \to Y$. For a fixed $x \in X$, consider the sequence $(f_n(x))_{n\in\mathbb{Z}_+}$ in Y. Fix $1 \geq \varepsilon > 0$, and let $N \in \mathbb{Z}_+$ such that $\overline{\rho}(f_m, f_n) < \varepsilon$ for every $m,n \geq N$. Then, for any $m,n \geq N$, note that

$$d(f_m(x), f_n(x)) < \sup_{x\in X}\{d(f_m(x), f_n(x))\} = \overline{\rho}(f_m, f_n) < \varepsilon,$$

and so $(f_n(x))$ is Cauchy. Since Y is complete, we know this sequence converges. Thus, for every $x \in X$, define $f(x)$ as the value $(f_n(x))$ converges to.

Finally, we show $f_n \to f$ in Y^X. Fix some $\varepsilon > 0$, then note there is some $N \in \mathbb{Z}_+$ such that $\overline{\rho}(f_m, f_n) < \varepsilon$ for every $m,n \geq N$. Fix n and let $m \to \infty$, then note that by the limit comparison theorem and continuity of the metric, $\overline{\rho}(f, f_n) \leq \varepsilon$. Thus $f_n \to f$.

If ε were chosen to be greater than 1 in either case, the bounded metric $\overline{\rho}$ would evaluate to 1, making the proceeding statements trivial.

Now, when defining Y^X for Y a metric space, let X be a topological space. The set Y^X is not different, but we may now consider subsets that we were unable to do before.

[7.1.13] Definition (*Space of Continuous and Bounded Functions*)

Let X be a topological space and Y be a metric space. The **space of continuous maps**, denoted $C(X,Y) \subseteq Y^X$, is given by

$$C(X,Y) = \{f \mid f : X \to Y \text{ is continuous}\}.$$

Similarly, the **space of bounded maps**, denoted $B(X,Y) \subseteq Y^X$, is given by

$$B(X,Y) = \{f \mid f : X \to Y \text{ is bounded}\}.$$

Recall that a map $f : X \to Y$, with Y a metric space, is said to be **bounded** if $f(X)$ is contained in a single ball.

[7.1.14] Theorem

Let X be a topological space and (Y,d) a metric space. Then the spaces $C(X,Y)$ and $B(X,Y)$ are closed.

Proof. Note these spaces are metric, so sequences probe limit points. We show that any sequences contained in these spaces converge inside the space.

Let (f_n) be a sequence of functions in $C(X,Y)$, meaning each $f_n : X \to Y$ is continuous, such that $f_n \to f$. Note that convergence in this space is uniform convergence, so f is automatically continuous and so $f \in C(X,Y)$. Since $f_n \to f$, for any $\varepsilon > 0$ there is some $N \in \mathbb{Z}_+$ such that $\overline{\rho}(f_n, f) < \varepsilon$. Then, for any $x \in X$, note that

$$\overline{d}(f_n(x), f(x)) < \overline{\rho}(f_n, f) < \varepsilon.$$

Note that ε is an upper bound for $\overline{d}(f_n(x), f(x))$ for *every* $x \in X$, independent of ε. Thus $f_n \to f$ uniformly, and so f is continuous, meaning $f \in C(X,Y)$ and so the space is closed.

Now let (f_n) be a sequence of bounded functions in $B(X,Y)$, meaning there is some $\varepsilon_n > 0$ for each $f_n : X \to Y$ such that $f_n(X) \subseteq B_{\varepsilon_n}(\mathbf{0})$. Note that for $\varepsilon = 1$, there is some $N \in \mathbb{Z}_+$ such that $d(f_n, f) < 1$ for every $n \geq N$. Note then that

$$d(\mathbf{0}, f) \leq d(\mathbf{0}, f_n) + d(f_n, f) \leq \varepsilon_n + 1$$

for every $n \geq N$. Thus $f \in B_{\varepsilon_N + 1}(\mathbf{0})$, meaning $f \in B(X,Y)$. Thus the space is closed, completing the proof.

[7.1.15] Theorem

Let X be a topological space and (Y,d) a *complete* metric space. Then the spaces $C(X,Y)$ and $B(X,Y)$ are *complete*.

[7.1.16] Definition (*Sup Metric*)

If (Y,d) is a metric space, one can define another metric, the **sup metric**, on the set $B(X,Y)$ of bounded functions $X \to Y$ by the equation

$$d_\infty(f,g) = \sup_{x\in X}\{d(f(x),g(x))\}.$$

Note that this metric is well defined since $f(X)\cup g(X)$ is bounded above, and so $d(f(x),g(x))$ is bounded above for all $x \in X$. In fact, the sup metric may be put on all other bounded sets. The sup metric is almost identical to the uniform metric, in which $\overline{\rho}(f,g) = \min\{d_\infty(f,g),1\}$.

[7.1.17] Theorem (*Metric Space Completion Theorem*)

Let (X,d) be a metric space. Then there is an isometric embedding of X into a complete metric space.

Proof. Let $x_0 \in X$ be fixed. For any $a \in X$, define the map

$$\varphi_a(x) = d(x,a) - d(x,x_0).$$

We show $\varphi_a \in B(X,\mathbb{R})$. Indeed, note that

$$\begin{aligned} d(x,a) &\le d(x,x_0) + d(x_0,a), \\ d(x,x_0) &\le d(x,a) + d(a,x_0). \end{aligned}$$

Then we have that

$$\Big(|\varphi_a(x)| = |d(x,a) - d(x,x_0)|\Big) \le |d(x,x_0) - d(x,a)| \le d(x,x_0) + d(x,a) \le d(x_0,a).$$

Thus $|\varphi_a(x)| \le |d(x_0,a)|$ for every $x \in X$, meaning φ_a is bounded and so $\varphi_a \in B(X,\mathbb{R})$. Finally, define the map

$$\Phi_a : X \to B(X,\mathbb{R}) \qquad \Phi_a(x) = \varphi_a(x).$$

Note that $B(X,\mathbb{R})$, where $\mathbb{R}$ takes the Euclidean metric, is complete with the sup metric. We show Φ_a is an isometry. Note that an isometry between metric spaces automatically constitutes an embedding, completing the proof.

To show Φ_a is an isometry, we show

$$\rho(\varphi_a,\varphi_b) = d(a,b).$$

For any $a,b \in X$, note that

$$\begin{aligned} \rho(\varphi_a,\varphi_b) &= \sup_{x\in X}\Big\{d(\varphi_a(x),\varphi_b(x))\Big\} \\ &= \sup_{x\in X}\Big\{\big|\varphi_a(x) - \varphi_b(x)\big|\Big\} \\ &= \sup_{x\in X}\Big\{|(d(x,a) - d(x,x_0)) - (d(x,b) - d(x,x_0))|\Big\} \\ &= \sup_{x\in X}\Big\{|d(x,a) - d(x,b)|\Big\} \\ &\le d(x,a) + d(x,b) \le d(a,b). \end{aligned}$$

Finally, note that this inequality cannot be strict. Take $x = a$, and note

$$|d(x,a) - d(x,b)| = |d(a,b)| \not< d(a,b).$$

❦

[7.1.18] DEFINITION (*Completion of a Metric Space*)

Let (X,d) be any metric space. Let $h: X \to Y$ be an isometric embedding into a complete metric space. Then the space $\overline{h(X)}$ is a complete metric space in Y, which we call the **completion**.

[7.1.19] EXAMPLE (*Completion of* $(\mathbb{Q},d)$)

A classic result is that the completion of $\mathbb{Q}$ under the metric $d(x,y) = |x-y|$ is $\mathbb{R}$ equipped with the standard Euclidean metric. Using the isometric embedding Φ into the complete space $B(\mathbb{Q},\mathbb{R})$, we map any $a \in \mathbb{Q}$ to the function $\varphi_a \in B(\mathbb{Q},\mathbb{R})$ defined by $\varphi_a(x) = |x-a| - |x|$ (where $x \in \mathbb{Q}$).

Consider, for example, a Cauchy sequence $(a_n) \subseteq \mathbb{Q}$ that approximates $\sqrt{2}$. In $\mathbb{Q}$, this sequence has no limit. However, the sequence of their images $\Phi(a_n) = \varphi_{a_n}$ is Cauchy in $B(\mathbb{Q},\mathbb{R})$. Because $B(\mathbb{Q},\mathbb{R})$ is complete, this sequence of functions must converge uniformly. Indeed, $\varphi_{a_n}(x) = |x-a_n| - |x|$ converges uniformly to the function $f(x) = |x-\sqrt{2}| - |x|$. This limit function f is a well-defined bounded function in $B(\mathbb{Q},\mathbb{R})$, but it is not in the image $\Phi(\mathbb{Q})$. The "hole" at $\sqrt{2}$ has thus materialized as a concrete boundary point of $\Phi(\mathbb{Q})$ in the function space.

Repeating this process for every Cauchy sequence in $\mathbb{Q}$ fills in all such holes. The closure of the image of $\mathbb{Q}$ is therefore exactly:

$$\overline{\Phi(\mathbb{Q})} = \left\{ f_r(x) = |x-r| - |x| : r \in \mathbb{R} \right\}.$$

This closed subspace of $B(\mathbb{Q},\mathbb{R})$ is, by definition, the completion of $\mathbb{Q}$. Note that this abstract space of functions is structurally identical to $\mathbb{R}$. The image of $\mathbb{R}$ under this isometry is precisely $\overline{\Phi(\mathbb{Q})}$, and so there is a surjective isometry between them. Since a surjective isometry is a perfect equivalence of metric spaces, we safely identify the completion of $\mathbb{Q}$ directly with $\mathbb{R}$.

[7.1.20] REMARK (*Intuition behind Completions*)

A completion effectively fills the holes in a metric space caused by Cauchy sequences that fail to converge. A Cauchy sequence consists of points that get arbitrarily close to one another; geometrically, it *ought* to converge to a limit. When a space is incomplete, it simply lacks the points that these sequences are aiming at. The completion process adjoins exactly these missing limit points. By embedding X isometrically into the humongous *complete* function space $B(X,\mathbb{R})$, we force the missing limit functions to exist. Thus we patch the holes of X and guarantee that every Cauchy sequence in the new space converges.

7.1.1 Exercises

[7.1.21] PROBLEM

Let (X,d) be a metric space. Suppose that, for some $\varepsilon > 0$, every ε-ball in X has compact closure. Show that X is complete.

Proof. Let (x_n) be any Cauchy sequence. There exists some $N \in \mathbb{Z}_+$ such that $d(x_m, x_n) < \varepsilon/2$ for every $m,n \geq N$. Then there exists finitely many closed ε-balls whose union contains $x_1, \dots, x_{N-1}$. Adjoin to this list of balls a single ball of radius ε containing all $x_N, x_{N+1}, \dots$. Let this list be $\{U_1, \dots, U_n\}$, and note that $\{\overline{U_1}, \dots, \overline{U_n}\}$ still covers the sequence. Since each closure of the finitely many sets is compact, the set $U = \bigcup_{k=1}^{n} U_k$ is compact. Since x_n is contained in the compact set U, and compactness is equivalent to sequential compactness in metrizable spaces, we have that x_n admits a convergent subsequence. Thus x_n is a convergent sequence, making X complete. ❦

[7.1.22] Problem

Let (X, d_X) and (Y, d_Y) be metric spaces, with Y complete. Let $A \subseteq X$. Show that, for any uniformly continuous $f : A \to Y$, there exists a *unique*, uniformly continuous extension to $g : \overline{A} \to Y$.

Proof. Let $x \in \overline{A} \setminus A$. Then there exists a sequence (x_n) in A such that $x_n \to x$. By uniform continuity of f, for any $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$d_X(x_1, x_2) < \delta \implies d_Y(f(x_1), f(x_2)) < \varepsilon$$

for every $x_1, x_2 \in A$. There is some $N \in \mathbb{Z}_+$ such that $d_X(x_m, x_n) < \delta$ for every $m, n \geq N$, and so $d_Y(f(x_m), f(x_n)) < \varepsilon$ for every $m, n \geq N$. Since ε is arbitrary, we have that $f(x_n)$ is Cauchy in the complete space Y. Thus $f(x_n) \to y_x$.

Now suppose (z_n) is some other sequence in A such that $z_n \to x$. Fix $\varepsilon > 0$. Once again, by uniform continuity, there exists a $\delta > 0$ such that

$$d_X(z_n, x_n) < \delta \implies d_Y(f(z_n), f(x_n)) < \varepsilon$$

for every $n \in \mathbb{Z}_+$. Since $z_n \to x$ and $x_n \to x$, note that $d_X(z_n, x_n) \to 0$, and so $d_Y(f(z_n), f(x_n)) \to 0$. Since $f(x_n) \to y_x$, we have that $f(z_n) \to y_x$.

Define a function $g : \overline{A} \to Y$ such that $g|_A = f$ and $g|_{\overline{A} \setminus A}(x) = y_x$, where y_x is chosen as per before. Note that any sequence tending to the same point x in $\overline{A} \setminus A$ has the same limit point y_x in the image, and so g is well-defined. We prove this function is uniformly continuous.

Fix $\varepsilon > 0$. By the uniform continuity of $g|_A = f$, there exists $\delta > 0$ such that

$$d_X(x_1, x_2) < \delta \implies d_Y(g(x_1), g(x_2)) < \varepsilon$$

for every $x_1, x_2 \in A$. We show this δ works for all points in $\overline{A}$. Indeed, suppose $u, v \in \overline{A}$ are such that $d_X(u, v) < \delta$. There exist sequences (u_n) and (v_n) in A such that $u_n \to u$ and $v_n \to v$. Because the metric d_X is continuous, $\lim_{n\to\infty} d_X(u_n, v_n) = d_X(u, v) < \delta$. Thus, there exists some $N \in \mathbb{Z}_+$ such that $d_X(u_n, v_n) < \delta$ for every $n \geq N$.

Applying the uniform continuity of $g|_A$ to these sequence elements, we find that $d_Y(g(u_n), g(v_n)) < \varepsilon$ for every $n \geq N$. By the construction of g, we know $g(u_n) \to g(u)$ and $g(v_n) \to g(v)$. Taking the limit as $n \to \infty$ and using the continuity of the metric d_Y, we obtain $d_Y(g(u), g(v)) \leq \varepsilon$. Since $\varepsilon > 0$ was arbitrary, we conclude that g is uniformly continuous on $\overline{A}$.

Recall that continuous functions preserve limits. g does this, since for every sequence $x_n \to x$, we define $g(x)$ such that $g(x_n) \to g(x)$. If $g|_{\overline{A} \setminus A}$ were defined in any other way, the resulting function would not be continuous. Thus g is the unique, uniformly continuous extension of f.

[7.1.23] Problem

Let (X, d) be a metric space. Recall that a map $f : X \to X$ is said to be a **contraction map** if there exists some $C < 1$ such that

$$d(f(x), f(y)) \leq C d(x, y).$$

Show that if X is complete, then any contraction map admits a *unique* fixed point.

Proof. Fix any $x_0 \in X$, then consider the sequence x_n defined by $x_n = f^{n-1}(x_0)$, where $f^0 = \mathrm{id}$.

We will show this sequence is Cauchy. Fix $\varepsilon > 0$. Note that $d(x_n, x_{n+1}) \leq C d(x_{n-1}, x_n)$ for any $n \in \mathbb{Z}_+$. Then note

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + \cdots + d(x_{n-1}, x_n) \\ &\leq C^m \underbrace{d(x_0, x_1)}_{:=r} + C^{m+1} d(x_0, x_1) + \cdots + C^{n-1} d(x_0, x_1) \\ &= r C^m \left(\frac{1 - C^{n-m}}{1 - C} \right) = r \left(\frac{C^m - C^n}{1 - C} \right). \end{aligned}$$

Note that $d(x_m, x_n)$ can be made arbitrarily small since $C < 1$ (and so $C^m - C^n \to 0$ as $m, n \to \infty$). Thus there is some $N \in \mathbb{Z}_+$ such that $d(x_m, x_n) < \varepsilon$ for every $m, n \geq N$. Thus the sequence is Cauchy and thus convergent, and so $x_n \to x^*$. We show x^* is our desired fixed point.

Note that f is C-Lipschitz and thus continuous, so $f(x_n) \to f(x^*)$. Of course, $d(f(x_n), x_n) \to 0$, and so they converge to the same limit. Thus $x^* = f(x^*)$.

Finally, we show x^* is the unique fixed point. Suppose $x' \in X$ is another fixed point. Then note $d(f(x^*), f(x')) = d(x^*, x') \le Cd(x^*, x')$. Then either $C = 0$, in which there is only one unique fixed point by inspection, or $d(x^*, x') = 0$ (and so $x^* = x'$), completing the proof.

7.2 Compactness in Metric Spaces

We have already shown a plethora of equivalences of compactness in metric spaces. Namely, being compact, sequentially compact, limit point compact, and having the finite intersection property are all equivalent in metric spaces.

Immediately, we note that compact metric spaces are complete. Indeed, compactness is equivalent with sequential compactness and so Cauchy sequences have convergent subsequences, so the space is complete. Complete metric spaces are not automatically compact, however (consider $\mathbb{R}$). We seek to find a condition on metric spaces that, together with completeness, is equivalent to compactness.

[7.2.1] Definition (*Total Boundedness*)

A metric space (X, d) is said to be **totally bounded** if, for any $\varepsilon > 0$, there exist finitely many ε-balls that cover X.

[7.2.2] Remark (*Total Boundedness and Boundedness*)

Note that any totally bounded space is bounded. Indeed, choose $\varepsilon = 1/2$ such that $\{B(x_k, 1/2)\}_{k=1}^n$ covers the entire space. From this, we have that $1 + \max\{d(x_i, x_j)\}_{i,j=0}^n$ is an upper bound on the diameter of the space, implying it is bounded. Bounded spaces are not totally bounded, though. $\mathbb{R}$ is bounded under the standard bounded metric $\overline{d}(x, y) = \min\{1, |x - y|\}$, but is not totally bounded (a countably infinite number of balls of radius ≤ 1 would be needed to cover $\mathbb{R}$).

Take the metric $d(x, y) = |x - y|$ and the space $\mathbb{R}$. It is complete but not totally bounded. Now consider $(-1, 1)$. It is totally bounded but not complete. Finally, note that $[-1, 1]$ is both complete and totally bounded.

The example from before illustrates an example of a crucial theorem. A metric space is compact if and only if it is complete *and* totally bounded. Note this resembles the theorem in $\mathbb{R}$ that a set is compact iff closed (i.e., complete) and bounded (i.e., totally bounded).

[7.2.3] Theorem (*Heine-Borel for Metric Spaces*)

A metric space is compact if and only if it is complete and totally bounded.

Proof.

($\Longrightarrow$): Suppose X is compact. Then X is sequentially compact, meaning any Cauchy sequence admits a convergent subsequence, making X complete. Moreover, for any $\varepsilon > 0$, the open cover $\{B(x, \varepsilon)\}_{x \in X}$ has a finite subcover $\{B(x_k, \varepsilon)\}_{k=1}^n$, making X totally bounded.

($\Longleftarrow$): Suppose X is complete and totally bounded. Let $(x_n) \subseteq X$ be any sequence. Since X is totally bounded, there exists a finite set of balls $\{B(x^k, 1)\}_{k=1}^n$ such that $x_n \in \bigcup_{k=1}^n B(x^k, 1)$ for every $n \in \mathbb{Z}_+$. By pigeonhole, there exists some $j \in \{1, \ldots, n\}$ such that infinitely many points of the sequence lie in $B(x^j, 1)$. Choose a single point in the ball from this sequence and label it x_{n_1}. Now note we may partition the ball $B(x^j, 1)$ by finitely many balls of radius $1/2$, find infinitely many points of x_n inside one of these balls, and label it x_{n_2} (such that $n_2 > n_1$). Repeat this for ball of radius $1/4$ and choosing x_{n_3} (such that $n_3 > n_2 > n_1$), and by induction, choose x_{n_k} through the ball of radius $1/2^{k-1}$ (such that n_k is greater than all previous choices). Since $1/2^{k-1} \to 0$, we have that there exists some $N \in \mathbb{Z}_+$ such that $d(x_{n_j}, x_{n_k}) < \varepsilon$ for every $j, k \ge N$. Thus any sequence x_n has a Cauchy subsequence, and since X is complete, it is convergent. Thus X is sequentially compact and thus compact.

The goal for the rest of this section is to characterize the compact subsets of $(C(X,\mathbb{R}^n),\rho)$, the metric space of continuous maps from a topological space X to $\mathbb{R}^n$ together with the sup metric. We use the result from before to identify compact subsets, culminating into what is known as the Arzela-Ascoli theorem. This theorem characterizes all compact subsets of $C(X,\mathbb{R}^n)$ with the sup metric as precisely the subsets that are closed, uniformly bounded, and equicontinuous.

[7.2.4] Definition (*Equicontinuity*)

Suppose X is a topological space and (Y,d) is a metric space. Let $\mathcal{F}$ be a family of functions $X \to Y$. For any $x_0 \in X$, we say the family $\mathcal{F}$ is **equicontinuous at** x_0 if, for any $\varepsilon > 0$, there exists a single neighborhood $U \subseteq X$ of x_0 such that, for every $x \in X$ and $f \in \mathcal{F}$,

$$d(f(x), f(x_0)) < \varepsilon.$$

If $\mathcal{F}$ is equicontinuous at all points in X, we say the family is simply equicontinuous.

Recall that for f to be continuous at x_0, we need that for any $\varepsilon > 0$ there exists an open neighborhood U of x_0 such that

$$d(f(x), f(x_0)) < \varepsilon$$

for every $x \in U$. Compare this to equicontinuity: for $\mathcal{F}$ to be equicontinuous at x_0, we need this relation to hold for every $f \in \mathcal{F}$ for only one U. Note that if X is metric, then the open neighborhood U of x_0 would just be a δ-neighborhood.

[7.2.5] Theorem

Suppose X is a topological space and (Y,d) a metric space. If $\mathcal{F} \subseteq C(X,Y)$ is totally bounded under the uniform metric $\overline{\rho}$, then it is equicontinuous under the metric d.

Proof. Suppose $\mathcal{F}$ is totally bounded. Fix $x_0 \in X$ and $\varepsilon \in (0,1)$. We seek to show there exists a neighborhood U of x_0 such that $d(f(x), f(x_0)) < \varepsilon$ for every $x \in U$ and $f \in \mathcal{F}$. Since $\varepsilon < 1$, we use d and $\overline{d}$ interchangeably, for they are equivalent.

Let $\delta = \varepsilon/3$. Since $\mathcal{F}$ is totally bounded, note there exists finitely many open balls $\{B_{\overline{\rho}}(f_k,\delta)\}_{k=1}^n$ from $C(X,Y)$ that covers $\mathcal{F}$. Note that each f_k is continuous, and so we can choose an open neighborhood U of x_0, we have that

$$d(f_k(x), f_k(x_0)) < \varepsilon$$

for every $x \in U$ and $k \in \{1,\dots,n\}$.

Fix any $f \in \mathcal{F}$. From the finite covering of $\mathcal{F}$, choose some f_k such that $\overline{\rho}(f,f_k) < \delta$. Finally, note

$$\begin{aligned} d(f(x), f(x_0)) &\le d(f(x), f_k(x)) + d(f_k(x), f_k(x_0)) + d(f_k(x_0), f(x_0)) \\ &\le 3\delta = 3\varepsilon/3 = \varepsilon. \end{aligned}$$

We get this relationship from the fact that $d(f(*), f_k(*)) < \overline{\rho}(f,f_k) < \delta$, as well as how we chose U. If $\varepsilon > 1$, then just apply the same procedure to some $\varepsilon' < 1 < \varepsilon$. Thus $\mathcal{F}$ is equicontinuous. ❦

[7.2.6] Theorem

Suppose X is a topological space and (Y,d) is a metric space. If $\mathcal{F} \subseteq C(X,Y)$ is equicontinuous under d, as well as X and Y being compact, we have that $\mathcal{F}$ is totally bounded under the uniform and sup metrics.

In sum, if $\mathcal{F} \subseteq C(X,Y)$ is totally bounded it is equicontinuous, and if it is equicontinuous (together with X,Y compact) it is totally bounded.

[7.2.7] Definition (*Pointwise Bounded*)

Suppose X is a topological space and (Y,d) is a metric space. A family of functions $\mathcal{F} \subseteq C(X,Y)$ is said to be **pointwise bounded** under d if, for each $a \in X$, the set $\{f(a) : f \in \mathcal{F}\}$ is bounded under d.

[7.2.8] Definition (*Uniformly Bounded*)

Suppose X is a topological space and (Y,d) is a metric space. A family of functions $\mathcal{F} \subseteq C(X,Y)$ is said to be **uniformly bounded** under d if there exists a point $y_0 \in Y$ and a real number $M > 0$ such that

$$d(f(x), y_0) \le M$$

for every $x \in X$ and $f \in \mathcal{F}$. If Y is a normed space such as $\mathbb{R}^n$, this is equivalent to saying there is some $M > 0$ such that $\|f(x)\| \le M$ for all $x \in X$ and $f \in \mathcal{F}$. Note that $\mathcal{F}$ is uniformly bounded under d if and only if $\mathcal{F}$ is bounded as a subset of $C(X,Y)$ under the sup metric ρ.

[7.2.9] Theorem (*Arzela-Ascoli Theorem*)

Let X be a compact space and $\mathbb{R}^n$ Euclidean space together with either the Euclidean or square metric. Give $C(X,\mathbb{R}^n)$ the corresponding sup metric. A subspace $\mathcal{F} \subseteq C(X,\mathbb{R}^n)$ has compact closure if and only if it is equicontinuous and pointwise bounded under $\mathbb{R}^n$'s metric.

[7.2.10] Theorem (*Arzela-Ascoli Theorem*)

Let X be a compact space and $\mathbb{R}^n$ Euclidean space together with either the Euclidean or square metric. Give $C(X,\mathbb{R}^n)$ the corresponding sup metric. A subspace $\mathcal{F} \subseteq C(X,\mathbb{R}^n)$ is compact if and only if it is closed, bounded under the sup metric, and equicontinuous under $\mathbb{R}^n$'s metric.

[7.2.11] Remark

The entirety of this section is dedicated to answering one question: what does a compact set look like in an infinite-dimensional function space? In finite dimensions ($\mathbb{R}^n$), Heine-Borel tells us that a set is compact if and only if it is closed and bounded. In infinite dimensions, this fails; a sequence of functions bounded between -1 and 1 can oscillate infinitely fast, never converging to anything.

To fix this, we look to metric spaces in general: a space is compact if and only if it is complete and totally bounded. We simply translate these two topological conditions into analytic conditions for functions. Because the uniform metric space $C(X,\mathbb{R}^n)$ is already complete, demanding that a subspace $\mathcal{F}$ be complete is exactly equivalent to demanding that $\mathcal{F}$ be **closed**. Thus our task boils down to finding an analytic condition on functions that is equivalent to total boundedness. Indeed, we proved lemmas that if a family of functions can be covered by finitely many ε-balls (total boundedness), it is forced to be **equicontinuous**, which prevents the functions from oscillating wildly. This is an equivalence (i.e., the converse is true) if the domain and codomain of these functions are compact.

We are not done yet, however. To characterize the compact subsets of $C(X,\mathbb{R}^n)$, we cannot directly use this equivalence between total boundedness and equicontinuity. Indeed, the codomain $\mathbb{R}^n$ is not compact. This is why **pointwise boundedness** is introduced.

Suppose $\mathcal{F}$ is pointwise bounded and equicontinuous and X is compact. We show these conditions force the family to be uniformly bounded. For each $a \in X$, equicontinuity gives an open neighborhood U_a of a such that $d(f(x), f(a)) < 1$ for every $x \in U_a$ and $f \in \mathcal{F}$. Since $\mathcal{F}$ is pointwise bounded, we have that $d(f(a), \mathbf{0}) \le M_a$ for each $a \in X$. Then for any $x \in U_a$ and any $f \in \mathcal{F}$, note that

$$d(f(x), \mathbf{0}) \le d(f(x), f(a)) + d(f(a), \mathbf{0}) < 1 + M_a.$$

Thus each $f \in \mathcal{F}$ is bounded in each U_a. Since $\{U_a\}_{a \in X}$ is an open cover for X and X is compact, there is a finite subcover $\{U_{a_1}, \ldots, U_{a_k}\}$. Simply choose the maximum of the upper bounds $(M_{a_j} + 1)$ from this finite collection, and this is what every function is bounded above by on all of X. Thus $\mathcal{F}$ is uniformly bounded.

This guarantees that the images of all functions in $\mathcal{F}$ lie inside a finite-radius closed ball in $\mathbb{R}^n$. By the standard Heine-Borel theorem for Euclidean space, this closed ball is compact. Thus, enforcing $\mathcal{F}$ to be pointwise bounded gives us an easy route to equate total boundedness and equicontinuity, since the images of these functions all lie in a compact codomain. Assembling the pieces: $\mathcal{F}$ is equicontinuous and pointwise bounded if and only if it is totally bounded, which together with the completeness of $C(X, \mathbb{R}^n)$ characterizes the subspaces with compact closure. This is the first version of the theorem.

The two versions of the Arzela-Ascoli theorem state the exact same mathematical fact. Note that in any metric space, a set A is compact if and only if it is closed and its closure $\overline{A}$ is compact.

The first version tells us exactly when the closure $\overline{\mathcal{F}}$ is compact (when $\mathcal{F}$ is equicontinuous and pointwise bounded). The second version simply adds the "closed" requirement to the checklist so that $\mathcal{F} = \overline{\mathcal{F}}$, meaning $\mathcal{F}$ itself is compact.

Furthermore, the first version requires the weaker condition of *pointwise* boundedness, while the second version requires boundedness under the sup metric, i.e. *uniform* boundedness. This is a free upgrade: if a family of functions is equicontinuous and pointwise bounded on a compact domain, it is automatically uniformly bounded, as shown above. The first version is often preferred in practice because checking pointwise bounds is algebraically much easier than proving uniform bounds.

7.3 Pointwise and Compact Convergence

[7.3.1] Definition (*Topology of Pointwise Convergence*)

Let X be a topological space. Given any $x \in X$ and open set U of the space Y, let

$$S(x, U) = \{f : f \in Y^X \text{ and } f(x) \in U\}.$$

The collection of all $S(x, U)$ forms a subbasis for a topology on Y^X called the **topology of pointwise convergence**.

Let's consider the basis elements for this topology, which are simply finite intersections of subbasis elements. For some $f : X \to Y$, a basis element about f would comprise functions g that are "close" to f at finitely many points. For example, let

$$B = \bigcap_{k=1}^{n} S(x_k, U_k)$$

be an arbitrary basis element about an arbitrary function $f \in Y^X$. For $g \in B$ to be true, we need that $g(x_k) \in U_k$ for each $k \in \{1, \ldots, n\}$. Of course, $f \in B$ as well, so this is the notion for which we say the functions are "close" to eachother.

Note this topology on Y^X is the exact same as the product topology. If we replace X with an indexing set Λ and denote a general element of Λ by λ, then the set $S(\lambda, U)$ simply comprises the functions $\mathbf{x} : \Lambda \to Y$ such that $\mathbf{x}(\lambda) = x_\lambda \in U$. This is precisely $\pi_\lambda^{-1}(U)$ of Y^Λ, and so subbasis elements agree with the product topology.

[7.3.2] Theorem

Let Y^X be a topological space with the topology of pointwise convergence (or otherwise the usual product topology). Then a sequence of functions f_n converges to a function f if and only if, for every $x \in X$, we have that $f_n(x) \to f(x)$. That is, $f_n \to f$ if and only if f_n converges pointwise to f.

[7.3.3] Definition (*Topology of Compact Convergence*)

Let X be a topological space and (Y,d) a metric space. Given some $f \in Y^X$, a compact subset $C \subseteq X$, and some $\varepsilon > 0$, we define

$$B_C(f,\varepsilon) = \left\{ g : \sup_{x\in C}\{d(f(x),g(x))\} < \varepsilon \right\}.$$

The collection of all $B_C(f,\varepsilon)$ forms a basis for a topology on Y^X called the **topology of compact convergence**.

The proof that this forms a basis for a topology is immediate. The idea behind this topology is that a basis element about f admits functions g that are "close" to f on an entire compact set, instead of finitely many points.

[7.3.4] Theorem

Let Y^X be a topological space with the topology of compact convergence. Then a sequence of functions f_n converges to f if and only if $f_n|_C \to f|_C$ uniformly on every compact $C \subseteq X$.

Proof. Recall that we say $f_n \to f$ uniformly on some $A \subseteq X$ if, for any $\varepsilon > 0$, there exists some $N \in \mathbb{Z}_+$ such that $d(f_n(x), f(x)) < \varepsilon$ for every $x \in A$. In other words, the choice of $N \in \mathbb{Z}_+$ must work for all points in A simultaneously, as opposed to at a single point like with uniform convergence. The notion of convergence induced by this topology is that, after fixing some compact set C, $f_n|_C \to f|_C$ if for any basis element $U = B_C(f,\varepsilon)$ (this step involves simply choosing any $\varepsilon > 0$), there is some $N \in \mathbb{Z}_+$ such that $f_n \in U$ for every $n \geq N$. If $f_n \in U$, then we have that $\sup_{x\in C} d(f(x), f_n(x)) < \varepsilon$. Thus $f_n|_C \to f|_C$ is precisely what it means for $f_n|_C \to f|_C$ uniformly.

[7.3.5] Theorem (*Topologies on Function Space*)

Let X be a space and (Y,d) a metric space. For the function space Y^X, one has the following inclusions of topologies:

$$(\text{uniform}) \supseteq (\text{compact convergence}) \supseteq (\text{pointwise convergence}).$$

If X is compact, the first two topologies coincide. If X is discrete, the last two topologies coincide.

8 Baire Spaces

Let's first consider what properties a set with empty interior has (that is, a set whose interior is the empty set). Many consequences come fairly quickly from this. For example, since the interior of a set A is the union of all open subsets of A, we conclude that a set with empty interior contains no open set besides the empty set. As another consequence, every open neighborhood about $x \in A$ must intersect $X \setminus A$, and so $X \setminus A$ is dense in X. There are two perspectives to be taken when considering a set with empty interior: the set itself containing no open set, and the rest of the space being dense.

[8.0.1] Definition (*Empty Interior*)

A set A is said to have empty interior based on the following three equivalent characterizations.

(1) $\mathrm{Int}(A) = \emptyset$.

(2) The only open set A contains is the empty set.

(3) $X \setminus A$ is dense.

[8.0.2] Definition (*Baire Space*)

A space X is said to be a **Baire space** if, for any countable collection $\{A_n\}$ of closed sets with empty interior, we have that $\bigcup A_n$ also has empty interior.

[8.0.3] Theorem (*Characterization of Baire Space*)

A space X is a Baire space if and only if, for any countable collection $\{U_n\}$ of open dense sets, we have that $\bigcap A_n$ is also dense.

[8.0.4] Theorem (*Baire Category Theorem*)

If X is a complete metric or compact Hausdorff space, then it is a Baire space.

Proof. Let $\{A_n\}$ be a countable collection of closed sets with empty interior. We show $\bigcup A_n$ also has closed interior. That is, for any nonempty set U_0 in X, we show there is some $x \in U_0$ such that $x \notin \bigcup A_n$.

Since A_0 has empty interior, note that A_0 cannot contain U_0, and so there is some $x \in U_0$ such that $x \notin A_0$. Note X is regular and A_0 is closed, so there exists an open neighborhood U_1 of x such that

$$\overline{U_1} \cap A_0 = \emptyset,$$
$$\overline{U_1} \subseteq U_0,$$
$$\mathrm{diam}(U_1) < 1 \quad \text{(if } X \text{ is metric)}.$$

We can continue this process by induction: any A_k cannot contain U_k, and so there is some $x \in U_k$ such that $x \notin A_k$. Then we can choose an open neighborhood U_{k+1} such that

$$\overline{U_{k+1}} \cap A_k = \emptyset,$$
$$\overline{U_{k+1}} \subseteq U_k,$$
$$\mathrm{diam}(U_{k+1}) < 1/(k+1) \quad \text{(if } X \text{ is metric)}.$$

We show $\bigcap \overline{U_n}$ is nonempty. If there is some $x \in \bigcap \overline{U_n}$, then we have that $x \in \overline{U_1} \subseteq U_0$ and $x \notin A_n$ for every n, completing the proof. In the case of X being compact Hausdorff, we note that the decreasing (ordered by inclusion) sequence of closed sets $\{\overline{U_n}\}$ has the finite intersection property, and so by compactness, $\bigcap \overline{U_n}$ is nonempty.

This result holds for complete metric spaces too. Indeed, since $\mathrm{diam}(U_n) \to 0$, we may form a sequence (x_n) such that $x_k \in C_k$ and note that it is Cauchy. By completeness, it converges to some x such that $x \in \overline{C_k}$ for every k, and so $\bigcap \overline{C_k}$ is nonempty.

[8.0.5] Theorem

Any open subset of a Baire space is a Baire space.

[8.0.6] Theorem

Let X be a space and (Y,d) a metric space. Suppose $f_n : X \to Y$ is a sequence of functions such that $f_n \to f$ pointwise. If X is Baire, then the set of points that f is continuous on is a dense set in X.