0 Prelude to Measure Theory

0.1 The Riemann Integral

We redefine the Riemann integral and basic results about it.

[0.1.1] Definition (*Partition, Refinement of Partition*)

Let $[a,b]$ be a closed interval in $\mathbb{R}$. A **partition** of the interval $[a,b]$ is a list of finitely many numbers $P = x_0, \dots, x_n$ such that

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b.$$

We say a partition P' is a **refinement** of P if P' is a sublist of P. That is, P' contains all of P and more.

We note that if $P = x_0, \dots, x_n$ is a partition of $[a,b]$, we have that

$$[a,b] = [x_0, x_1] \cup \cdots \cup [x_{n-1}, x_n].$$

A refinement $P' = x_0, x_{0.5}, x_1, \dots, x_{n-0.5}, x_n$ would give a finer partition of $[a,b]$ as such:

$$[a,b] = [x_0, x_{0.5}] \cup \cdots \cup [x_{n-0.5}, x_n].$$

[0.1.2] Definition (*Lower and Upper Riemann Sum*)

Suppose $f : [a,b] \to \mathbb{R}$ is a bounded function together with a partition P of $[a,b]$. We define the lower and upper Riemann sums of f on $[a,b]$ as such.

$$L(f, P, [a,b]) = \sum_{k=1}^{n} (x_k - x_{k-1}) \inf_{[x_{k-1}, x_k]} f,$$

$$U(f, P, [a,b]) = \sum_{k=1}^{n} (x_k - x_{k-1}) \sup_{[x_{k-1}, x_k]} f.$$

[0.1.3] Theorem (*Lower Riemann Sums Smaller Than Upper Riemann Sums*)

Let $f : [a,b] \to \mathbb{R}$ be any bounded function together with any two partitions P, P' of $[a,b]$. Then

$$L(f, P, [a,b]) \leq U(f, P, [a,b]).$$

[0.1.4] Definition (*Lower and Upper Riemann Integral*)

Suppose $f : [a,b] \to \mathbb{R}$ is a bounded function. The **lower Riemann integral** and **upper Riemann integral** of f are defined by

$$L(f, [a,b]) = \sup_P L(f, P, [a,b]),$$

$$U(f, [a,b]) = \inf_P U(f, P, [a,b]).$$

That is, the lower Riemann integral is the supremum of the lower Riemann sums over all possible partitions of $[a,b]$. Conversely, the upper Riemann integral is the infimum of the upper Riemann sums over all possible partitions of $[a,b]$. We take the *supremum* for the lower Riemann integral since the lower Riemann sums are always an underapproximation of the area under f (and refining makes it more precise), so taking the supremum over all partitions gives us the best approximation for the area under f. Vice-versa is also true.

[0.1.5] Theorem

The lower Riemann integral is less than or equal to the upper Riemann integral.

[0.1.6] Definition (*Riemann Integral*)

A bounded function $f : [a,b] \to \mathbb{R}$ is said to be **Riemann integrable** if its lower Riemann integral equals its upper Riemann integral. We define it by

$$\int_a^b f(x)\,\mathrm{d}x = L(f,[a,b]) = U(f,[a,b]).$$

[0.1.7] Theorem (*Continuous Function is Riemann Integrable*)

Suppose $f : [a,b] \to \mathbb{R}$ is continuous. Then it is Riemann integrable.

Proof. Recall that a continuous map from a compact set $[a,b]$ into $\mathbb{R}$ is uniformly continuous. Let $\varepsilon > 0$. Then there is some $\delta > 0$ such that

$$d(x_1,x_2) < \delta \implies d(f(x_1), f(x_2)) < \varepsilon.$$

Let $n \in \mathbb{Z}_+$ be defined such that $(b-a)/n < \delta$, and form a partition $P = x_0, \ldots, x_n$ of $[a,b]$ where $x_k - x_{k-1} = (b-a)/n$. Then note

$$\begin{aligned} U(f,[a,b]) - L(f,[a,b]) &\le U(f,P,[a,b]) - L(f,P,[a,b]) \\ &= \frac{b-a}{n} \sum_{k=1}^{n} \left(\sup_{[x_{k-1},x_k]} f - \inf_{[x_{k-1},x_k]} f \right) \\ &\le (b-a)\varepsilon. \end{aligned}$$

Note that $U(f,[a,b]) \le L(f,[a,b]) + (b-a)\varepsilon \le L(f,[a,b])$. Since the lower Riemann integral is always less than or equal than the upper Riemann integral, the two integrals are forced to be equal. ❦

[0.1.8] Theorem (*ML Inequality*)

Suppose $f : [a,b] \to \mathbb{R}$ is Riemann integrable. Then

$$(b-a)\inf_{[a,b]} f \le \int_a^b f(x)\,\mathrm{d}x \le (b-a)\sup_{[a,b]} f.$$

Proof. Let $P = a, b$ be the trivial partition. Then note

$$(b-a)\inf_{[a,b]} f = L(f,P,[a,b]) \le L(f,P,[a,b]) = \int_a^b f(x)\,\mathrm{d}x.$$

The second inequality follows immediately. ❦

0.2 Deficiencies of the Riemann Integral

We will discuss four problems with the Riemann integral.

(1) Riemann integration does not handle unbounded functions.

(2) Riemann integration does not handle functions with many discontinuities.

(3) Riemann integration does not work with limits of sequences of functions.

(4) The Riemann integral doesn't form a complete space when considering spaces of functions.

[0.2.1] Example (*Riemann Integration Against Unbounded Functions*)

Consider the function $f:[0,1]\to\mathbb{R}$ defined by

$$f(x)=\begin{cases}1/\sqrt{x} & 0<x\le 1,\\ 0 & x=0.\end{cases}$$

For any partition $P=x_0,\dots,x_n$ of $[0,1]$, note that $\sup_{[x_0,x_1]} f=\infty$, and so $U(f,[a,b])=\infty$ (thus $\int_0^1 f(x)\,\mathrm{d}x$ is undefined). But note

$$\lim_{a\to 0^+}\int_a^1 f(x)\,\mathrm{d}x=\lim_{a\to 0^+}[2\sqrt{x}]_{x=a}^1=\lim_{a\to 0^+}(2-2\sqrt{a})=2,$$

so we'd desire the integral to equal 2. This function is well-behaved and has the classic notion of "area under the curve," so it makes no sense for the integral to diverge.

[0.2.2] Example (*Riemann Integration Against Discontinuous Functions*)

Consider the function $f:[0,1]\to\mathbb{R}$ defined by $f(x)=1_{\mathbb{Q}}$. For any closed interval $[a,b]\subseteq[0,1]$, note $\inf_{[a,b]} f=0$ and $\sup_{[a,b]} f=1$, since any interval in $\mathbb{R}$ contains an irrational and rational number. Thus $L(f,[a,b])=0$ and $U(f,[a,b])=1$ for all partitions, meaning f is not Riemann integrable. But note that $\mathbb{Q}$ is countable and $[0,1]\setminus\mathbb{Q}$ is uncountable, so f is zero *far* more often than it is 1. With this in mind, one might want to define the integral of f to be 0, which is not possible with the Riemann integral.

[0.2.3] Example (*Riemann Integration Against Sequences of Functions*)

Let $r_1,r_2,\dots$ be an injective sequence $\mathbb{Z}_+\to\mathbb{Q}$ that contains all rational numbers in $[0,1]$. For $k\in\mathbb{Z}_+$, define $f_k:[0,1]\to\mathbb{R}$ by

$$f_k(x)=\begin{cases}1 & \text{if } x\in\{r_1,\dots,r_k\},\\ 0 & \text{otherwise.}\end{cases}$$

Recall that a function is Riemann integrable if its set of discontinuities is, at most, measure zero. The set of discontinuities for each f_k is finite, and so this function is Riemann integrable. Indeed, $\int_0^1 f_k=0$ for each $k\in\mathbb{Z}_+$. But note that $f_k\to f$ pointwise, with $f=1_{\mathbb{Q}}$, which we have established is not Riemann integrable. Thus

$$\lim_{n\to\infty}\int_0^1 f_n(x)\,\mathrm{d}x\neq\int_0^1\lim_{n\to\infty} f_n(x)\,\mathrm{d}x.$$

[0.2.4] Example (*Space of Continuous Functions Together with Riemann Integral Norm*)

Consider the space of continuous functions $f:[0,1]\to\mathbb{R}$, denoted $C([0,1])$, together with the norm

$$\|f-g\|_1=\int_0^1|f(x)-g(x)|\,\mathrm{d}x.$$

Ideally, we want this space to be complete with respect to the norm—we show this is not the case. Actually, we don't show it. I'm tired.

[0.2.5] Theorem (*Interchanging Limit and Integral with Riemann Integration*)

Suppose $\{f_k\}$ is a sequence of Riemann integrable functions $f_k : [a,b] \to \mathbb{R}$ such that $f_k \to f$ pointwise. If f is Riemann integrable on $[a,b]$, then

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{n\to\infty} \int_a^b f_k(x)\,\mathrm{d}x.$$

This seems like it'd resolve problems the Riemann integral has with sequences of functions, but this is also not true. First of all, the proof for this theorem is unwieldy, and it immediately suggests something about our theory is not up to par. Considering the theorem further, we need that the limit function is Riemann integrable, which is undesirable. After all, we would want a sequence of Riemann integrable functions to converge pointwise to a Riemann integrable function, right? This mirrors the fact that a sequence of continuous functions need not converge pointwise to a continuous function. To fix this, we made a stronger notion of convergence (uniform convergence) that forced the limit function to be continuous.

It is true that we can repair this theorem by using uniform convergence. Indeed, if $f_n \to f$ uniformly, then

$$\lim_{n\to\infty} \int_a^b f_n = \int_a^b \lim_{n\to\infty} f_n.$$

Uniform convergence is quite a strong restriction to place on a sequence of functions and is undesirable. All the problems posed by trying to swap limit and integral, together with the aforementioned problems, motivate us to throw out Riemann integration and discover a new, better integration theory. This is the purpose of measure theory.

0.3 Outer Measure on the Reals

[0.3.1] Definition (*Length of Open Interval*)

We denote the length of an open interval by

$$\ell(I) = \begin{cases} b-a & I=(a,b), \\ 0 & I=\emptyset, \\ \infty & I=(-\infty,a),(a,\infty),\ \text{or}\ (-\infty,\infty). \end{cases}$$

If we fully cover a set A with open intervals in the most conservative way possible, the length of all intervals in the cover would seem to be a good notion of "size" for A. And so it is.

[0.3.2] Definition (*Outer Measure*)

The **outer measure** of a set $A \subseteq \mathbb{R}$ is defined by

$$\mu^*(A) = \inf\left\{\sum_{k=1}^{\infty} \ell(I_k) : I_1, I_2, \ldots \text{ are open intervals such that } A \subseteq \bigcup_{k=1}^{\infty} I_k\right\}.$$

The measure takes ∞ as a value—that is, $\mu^* : \mathbb{R} \to [0,\infty]$. Thus, for example, $\mu^*(\mathbb{R}) = \infty$. The outer measure has many desirable properties.

[0.3.3] Theorem (*Countable Sets Are Measure Zero*)

Suppose $A \subseteq \mathbb{R}$ is countable. Then $\mu^*(A) = 0$.

Proof. Suppose $A = \{a_1, a_2, \ldots\}$ is a countable subset of $\mathbb{R}$. Fix some $\varepsilon > 0$. For each $k \in \mathbb{Z}_+$, define $I_k = (a_k - \varepsilon/2^k, a_k + \varepsilon/2^k)$. Then note

$$\mu^*(A) \le \sum_{k=1}^{\infty} \ell(I_k) = 2\varepsilon \sum_{k=1}^{\infty} \frac{1}{2^k} = 2\varepsilon.$$

Since ε is arbitrary, we have that $\mu^*(A) = 0$.

[0.3.4] Theorem (*Outer Measure Preserves Set Inclusion Order*)

Suppose A, B are subsets of $\mathbb{R}$ such that $A \subseteq B$. Then $\mu^*(A) \le \mu^*(B)$.

[0.3.5] Theorem (*Outer Measure is Translation Invariant*)

Let A be any subset of $\mathbb{R}$ and t any arbitrary number in $\mathbb{R}$. Then $\mu^*(A) = \mu^*(t + A)$.

Let $(1,4)$ and $(3,5)$ be open intervals, whose union is $(1,5)$. Then note that

$$\mu^*((1,4) \cup (3,5)) < \mu^*((1,4)) + \mu^*((3,5)),$$

because the LHS is 4 and the RHS is 5. The intuition behind this is that the RHS counts $(3,4)$ twice, whereas LHS counts it only once. We will generalize this to show the outer measure on $\mathbb{R}$ is **countably subadditive**.

[0.3.6] Theorem (*Outer Measure is Countably Subadditive*)

Suppose $A_1, A_2, \dots$ is a sequence of subsets of $\mathbb{R}$. Then

$$\mu^*\left(\bigcup_{k=1}^{\infty} A_k\right) \le \sum_{k=1}^{\infty} \mu^*(A_k).$$

Proof. If $\mu^*(A_k) = \infty$ for any $k \in \mathbb{Z}_+$, the result would be immediate, so suppose $\mu^*(A_k) < \infty$ for every $k \in \mathbb{Z}_+$. Fix $\varepsilon > 0$. For each $k \in \mathbb{Z}_+$, define a sequence of open intervals $A_{1,k}, A_{2,k}, \dots$ that cover A_k in such a way that

$$\sum_{j=1}^{\infty} \ell(A_{j,k}) \le \frac{\varepsilon}{2^k} + \mu^*(A_k).$$

Simply consider the collection $\{A_{j,k}\}_{j,k \in \mathbb{Z}_+}$. Note the sum over these indices yields

$$\sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \ell(A_{j,k}) \le \varepsilon + \sum_{k=1}^{\infty} \mu^*(A_k).$$

Note that this collection also covers $\bigcup_{k=1}^{\infty} A_k$ by construction, and so

$$\mu^*\left(\bigcup_{k=1}^{\infty} A_k\right) \le \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \ell(A_{j,k}) \le \varepsilon + \sum_{k=1}^{\infty} \mu^*(A_k).$$

Since $\varepsilon > 0$ is arbitrary, we have achieved that

$$\mu^*\left(\bigcup_{k=1}^{\infty} A_k\right) \le \sum_{k=1}^{\infty} \mu^*(A_k).$$

❦

Countable subadditivity implies finite subadditivity, since we can let $A_k = \emptyset$ for every $k > n$, then we have

$$\mu^*\left(\sum_{k=1}^{n} A_k\right) \le \sum_{k=1}^{n} \mu^*(A_k).$$

[0.3.7] Theorem (*Outer Measure of Closed Intervals*)

$\mu^*([a,b]) = b - a$.

Proof. Note that, for any $\varepsilon > 0$, the sequence of open sets $(a-\varepsilon, b+\varepsilon), \emptyset, \emptyset, \dots$ forms an open cover for $[a,b]$, and so

$$\mu^*([a,b]) \le (b-a) + 2\varepsilon \le b - a.$$

We simply show that $\mu^*([a,b]) \ge b - a$. Let $\{I_n\}_{n \in \mathbb{Z}_+}$ be *any* countable open cover for $[a,b]$. By compactness of $[a,b]$, there exists a finite subcover $\{I_k\}_{k=1}^{n}$. We prove by induction that

$$\sum_{k=1}^{n} \mu^*(I_k) \ge b - a,$$

and so $\sum_{k=1}^{\infty} \mu^*(I_k) \ge \sum_{k=1}^{n} \mu^*(I_k) \ge b - a$, meaning $\mu^*([a,b]) = b - a$.

If $[a,b] \subseteq I_1$, then $\mu^*(I_1) \ge b - a$, of course. Now suppose we know that, if $[a,b] = I_1 \cup I_2 \cup \cdots \cup I_n$, then

$$\sum_{k=1}^{n} \mu^*(I_k) \ge b - a.$$

Suppose $[a,b] \subseteq I_1 \cup \cdots \cup I_{n+1}$. Write $I_{n+1} = (c,d)$, and without loss of generality suppose $b \in I_{n+1}$. Note that if $c \le a$, then $\mu^*(I_{n+1}) \ge b + a$, so we are done. Thus suppose $a < c < b < d$, and so $[a,c] \subseteq I_1 \cup \cdots \cup I_n$. By inductive hypothesis, note that

$$\sum_{k=1}^{n+1} \mu^*(I_k) \ge (c-a) + \mu^*(I_{n+1}) \ge (c-a) + (d-c) = d - a \ge b - a.$$

Thus $\mu^*([a,b]) = b - a$.

❦

[0.3.8] Corollary (*Intervals in $\mathbb{R}$ are Uncountable*)

Every interval in $\mathbb{R}$ that contains at least two distinct elements is uncountable.

Proof. Suppose I is an interval containing $a, b \in \mathbb{R}$ with $a < b$. Then

$$\mu^*(I) \geq \mu^*([a,b]) = b - a \neq 0.$$

Countable sets are measure zero, and so I is not countable. ❦

The outer measure does have some unpleasant properties, however. We would expect that, for any disjoint sets A, B, the outer measure of $A \cup B$ is equal to the sum of the outer measure of A and B respectively. Unfortunately, this is not true in general.

[0.3.9] Theorem (*Nonadditivity of Outer Measure*)

There exist disjoint subsets $A, B \subseteq \mathbb{R}$ such that $\mu^*(A \cup B) \neq \mu^*(A) + \mu^*(B)$.

Proof. Consider the set $[-1,1]$. For any $a, b \in [-1,1]$, define the equivalence relation

$$a \sim b \iff a - b \in \mathbb{Q}.$$

Then the equivalence relation $\sim$ partitions $[-1,1]$ in such a way that

$$[-1,1] = \bigcup_{a \in [-1,1]} [a]_\sim.$$

For each coset, choose exactly one point, then let V be the set of all these points. That is, $V \cap [a]_\sim$ has a single, unique element for each $a \in [-1,1]$.

Let $r_1, r_2, \ldots$ be a sequence of points such that $[-2,2] \cap \mathbb{Q} = \{r_1, r_2, \ldots\}$. Then note

$$[-1,1] \subseteq \bigcup_{k=1}^{\infty} (r_k + V).$$

Indeed, for any $a \in [-1,1]$, note there is some $v \in V \cap [a]_\sim$. Then $a - v \in \mathbb{Q}$, meaning there is some $k \in \mathbb{Z}_+$ such that $a = r_k + v$, meaning $a \in r_k + V$ as desired. By order properties and countable subadditivity of the outer measure, note

$$2 = \mu^*([-1,1]) \leq \mu^*\left(\bigcup_{k=1}^{\infty} r_k + V\right) \leq \sum_{k=1}^{\infty} \mu^*(r_k + V) = \sum_{k=1}^{\infty} \mu^*(V).$$

Thus $\mu^*(V) > 0$.

Let $n \in \mathbb{Z}_+$. Note that

$$\bigcup_{k=1}^{n} (r_k + V) \subseteq [-3,3],$$

since $V \subseteq [-1,1]$ and $r_k \in [-2,2]$ for each k. Again, by order properties of the outer measure, note

$$\mu^*\left(\bigcup_{k=1}^{n} (r_k + V)\right) \leq \sum_{k=1}^{n} \mu^*(V) \leq 6.$$

But also note that

$$\sum_{k=1}^{n} \mu^*(V) = n\mu^*(V).$$

Simply choose $n \in \mathbb{Z}_+$ such that $n\mu^*(V) > 6$, and so

$$\mu^*\left(\bigcup_{k=1}^{n} (r_k + V)\right) \leq \sum_{k=1}^{n} \mu^*(r_k + V).$$

Note each $r_k + V$ is disjoint. Since $\mu^*(A_1 \cup \cdots \cup A_n) \neq \mu^*(A_1) + \cdots + \mu^*(A_n)$ in general for disjoint $A_1, \ldots, A_n$, by induction we have $\mu^*(A \cup B) \neq \mu^*(A) + \mu^*(B)$ in general. ❦

[0.3.10] Remark (*Intuition behind Disproof of Additivity of Outer Measure*)

We start by taking $[-1,1]$ and forming equivalence classes that group numbers that differ by a rational together. From each equivalence class, we choose one point, and we take the set of these points to be V. Crucially, note that no elements of V differ by a rational, otherwise two elements of V would be in the same equivalence class, contrary to construction. We then make copies of the set, shifting V by all the rationals in $[-2,2]$, written as a sequence $r_1, r_2, \ldots$. The translates $r_1 + V, r_2 + V, \ldots$ are all disjoint from eachother, since if they overlapped, there would be two numbers in V that differed by a rational. We will refer to $r_k + V$ as a "copy" of V, since it is still isomorphic to V.

If we take infinitely many of these copies and union them together, we note it forms a cover for $[-1,1]$. Thus the outer measure of this union is greater than 2, establishing the outer measure of V is nonzero.

Now, we look at finitely many copies of V, say n of them. The union of n many copies of V lie in $[-3,3]$, and so the outer measure of this union is bounded above by 6. But the sum of n many outer measures of V is given by n multiplied by the outer measure of V, which is nonzero. This sum grows without bound, so there exists some n such that $n\mu^*(V) > 6$, completing the disproof.

1 Measures