1 Set Theory and Categories

1.1 Equivalence Relations

[1.1.1] Definition (*Equivalence Relation*)

An **equivalence relation** on a set A is any relation $\sim$ satisfying the three following properties:

(1) **Reflexivity.** $a \sim a$ for every $a \in A$.

(2) **Symmetry.** $a \sim b$ if and only if $b \sim a$ for every $a, b \in A$.

(3) **Transitivity.** If $a \sim b$ and $b \sim c$, then $a \sim c$ for every $a, b, c \in A$.

[1.1.2] Remark (*Partitions, Equivalence Classes*)

Equivalence relations **partition** a set S, where a partition of a set S is a set of nonempty, disjoint subsets of S whose union is S itself. For partitions induced by equivalence relations, we say each subset is an **equivalence class**, defined by

$$[a]_\sim = \{b : b \sim a\}.$$

[1.1.3] Definition (*Quotient by Equivalence Relation*)

The **quotient** of a set S with respect to an equivalence relation $\sim$ is the set of all equivalence classes induced by $\sim$. That is,

$$S/\sim = \{[a]_\sim : a \in S\}.$$

Note. Sets are deduplicated by definition, so redundant equivalence classes are culled.

[1.1.4] Example (*Integers Modulo 2*)

Take $S = \mathbb{Z}$ and $\sim$ to be such that

$$a \sim b \iff a - b \text{ is even } (a \equiv b \pmod 2).$$

Then $\mathbb{Z}/\sim = \{[0]_\sim, [1]_\sim\}$, with

$$[0]_\sim = \{z \in \mathbb{Z} : z \equiv 0 \pmod 2\} \qquad [1]_\sim = \{z \in \mathbb{Z} : z \equiv 1 \pmod 2\}.$$

A word is in order about equivalence classes of numbers modulo n, which will be discussed in the near future.

1.2 Disjoint Unions and Cartesian Products

A word about disjoint unions and Cartesian products is in order.

[1.2.1] Definition (*Cartesian Product*)

The **Cartesian product** of two sets A, B is defined by

$$A \times B = \{(a,b) : a \in A, b \in B\}.$$

[1.2.2] Remark (*Cartesian Product is not Canonical*)

For two sets V_1, V_2, their Cartesian product is uniquely determined—it is just a set of all the pairs of elements in V_1 and V_2. But consider the Cartesian product of more than two sets, say $V_1 \times V_2 \times V_3$. We can interpret it in two ways:

$$V_1 \times (V_2 \times V_3) = \{(v_1, (v_2, v_3)) : v_k \in V_k\},$$

$$(V_1 \times V_2) \times V_3 = \{((v_1, v_2), v_3) : v_k \in V_k\}.$$

Effectively, the Cartesian product is not associative. Thus $V_1 \times \cdots \times V_n$ is ill-defined as a set since a choice needs to be made.

[1.2.3] Definition (*Disjoint Union of Sets*)

Suppose S, T are two sets. Let S', T' be such that $S \cong S'$, $T \cong T'$, and $S' \cap T' = \emptyset$. Then the **disjoint union** of S, T is $S \amalg T = S' \cup T'$.

[1.2.4] Remark (*Disjoint Union is not Canonical*)

Of course, the disjoint set union is not canonical—the isomorphic sets used in the expression are chosen arbitrarily. One can let $S' = S \times \{0\}$ and $T' = T \times \{1\}$ for a convenient choice, but, of course, this is a choice.

We finish by noting that the Cartesian product (of three or more sets) and disjoint union of sets are not canonically chosen and are technically undefined as unique sets. However, any choices made in their construction lead to isomorphic candidates, meaning that the results of these operations are well-defined up to isomorphism. The main feature of constructions like products and disjoint unions is not really "what elements they contain" but rather "their relationship with all other sets".

1.3 Functions of Sets

[1.3.1] Definition (*Function, Graph of Function*)

A **function** $f : A \to B$ is a map between sets in which each element $a \in A$ is mapped to exactly one $f(a) \in B$. The **graph** of a function is defined by

$$\Gamma_f = \{(a,b) \in A \times B : b = f(a)\} = \{(a, f(a)) : a \in A\}.$$

As notation, note $a \mapsto f(a)$ means a "maps to" $f(a)$, and

$$A \xrightarrow{f} B$$

diagrammatically notes A is mapped to B by the function f.

Suppose $f : A \to B$ and $g : B \to C$. The composition $(g \circ f) : A \to C$ can be drawn by a commutative diagram.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "g\circ f"'] & B \arrow[r, "g"] & C \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "g\circ f"'] & B \arrow[d, "g"] \\ & C \end{tikzcd}$$

[1.3.2] Definition (*Commutative Diagram*)

A diagram is said to **commute** if, when going through any path prescribed by the diagram between two endpoints, the result is the same.

Suppose $f : A \to B, g : B \to C, h : C \to D$.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "(g\circ f)"'] & B \arrow[r, "g"] \arrow[rr, bend left, "(h\circ g)"] & C \arrow[r, "h"] & D \end{tikzcd}$$

Note the diagram is commutative, and accordingly, $h \circ (g \circ f) = (h \circ g) \circ f$. The associativity of function compositions is trivial and left without proof.

[1.3.3] Definition (*Injectivity and Surjectivity*)

A function $f : A \to B$ is:

(1) **injective** if, for every $a, a' \in A$ such that $a \neq a'$, it follows that $f(a) \neq f(a')$. Equivalently, for every $a, a' \in A$ such that $f(a) = f(a')$, it follows that $a = a'$.

(2) **surjective** if, for every $b \in B$, there exists some $a \in A$ such that $f(a) = b$. Equivalently, if $\mathrm{im}(f) = B$.

Injections and surjections are noted by $\hookrightarrow$ and $\twoheadrightarrow$ respectively, a homage to the notation of monomorphisms and epimorphisms (an equivalent categorization in the category of sets, Set).

If $f : A \to B$ is both injective and surjective, it is indeed bijective, and accordingly A and B are isomorphic as sets. We say $f : A \xrightarrow{\sim} B$, or $A \cong B$.

Suppose $f : A \to B$ is bijective. Then, we are justified in "flipping" the diagram that defines f to produce $g : B \to A$ such that $a = g(b)$ whenever $b = f(a)$. Diagrammatically,

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "\mathrm{id}_A"'] & B \arrow[r, "g"] & A \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "g"'] \arrow[rr, bend left, "\mathrm{id}_B"] & A \arrow[r, "f"'] & B \end{tikzcd}$$

commutes, so $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$. The first diagram says g is a left-inverse of f (as reference to g being left to f in composition notation, which is read right-to-left), and conversely the second diagram says g is a right-inverse of f. Indeed, we simply say g is the inverse of f, for which we write $g = f^{-1}$ since two-sided inverses are unique (this is trivial and left without proof).

[1.3.4] Theorem (*Injectivity, Surjectivity, Bijectivity Characterized by Invertibility*)

Suppose $f : A \to B$ with $A \neq \emptyset$. Then

(1) f has a left-inverse if and only if it is injective.

(2) f has a right-inverse if and only if it is surjective.

(3) f has a (unique) two-sided inverse if and only if it is bijective.

Proof. This is trivial and left without proof. ❦

1.4 Monomorphisms, Epimorphisms

[1.4.1] Definition (*Monomorphism*)

A function of sets $f : A \to B$ is a **monomorphism** if, for every set Z and functions $\alpha', \alpha'' : Z \to A$ such that $f \circ \alpha' = f \circ \alpha''$, it follows that $\alpha' = \alpha''$.

[1.4.2] Theorem (*Monomorphisms On Set-Functions Are Equivalent to Injections*)

Suppose $f : A \to B$ is a function of sets. Then f is injective if and only if it is a monomorphism.

Proof. $(\Longrightarrow)$: Suppose f is injective. Let Z be an arbitrary set and let $\alpha', \alpha'' : Z \to A$ be functions such that

$$f \circ \alpha' = f \circ \alpha''.$$

Note that f has a left-inverse g, so we may apply it both sides and make use of associativity:

$$(g \circ f) \circ \alpha' = (g \circ f) \circ \alpha'' \implies \mathrm{id}_A \circ \alpha' = \mathrm{id}_A \circ \alpha'' \implies \alpha' = \alpha''.$$

Thus f is monomorphic.

$(\Longleftarrow)$: Suppose f is monomorphic. Let Z be an arbitrary set and let $\alpha', \alpha'' : Z \to A$ be functions such that

$$f \circ \alpha' = f \circ \alpha''.$$

Then it follows that $\alpha = \alpha'$. For any $z \in Z$, we have that $\alpha'(z) = a_1 \in A$ and $\alpha''(z) = a_2 \in A$. Then we have, by equality of the two compositions, that

$$f(a_1) = f(a_2) \implies a_1 = a_2$$

for all arbitrary pairs of elements in A. Thus f is injective. ❦

[1.4.3] Definition (*Epimorphism*)

A function of sets $f : A \to B$ is a **epimorphism** if, for every set Z and functions $\beta', \beta'' : B \to Z$ such that $\beta' \circ f = \beta'' \circ f$, it follows that $\beta' = \beta''$.

[1.4.4] Theorem (*Epimorphisms On Set-Functions Are Equivalent to Surjections*)

Suppose $f : A \to B$ is a function of sets. Then f is surjective if and only if it is an epimorphism.

Proof. $(\Longrightarrow)$: Suppose f is surjective. Let Z be an arbitrary set and let $\beta', \beta'' : B \to Z$ be functions such that

$$\beta' \circ f = \beta'' \circ f.$$

Note that f has a right-inverse g, so we may apply it both sides and make use of associativity:

$$\beta' \circ (f \circ g) = \beta'' \circ (f \circ g) \Longrightarrow \beta' \circ \mathrm{id}_B = \beta'' \circ \mathrm{id}_B \Longrightarrow \beta' = \beta''.$$

Thus f is epimorphic.

$(\Longleftarrow)$: Suppose f is not surjective. Let Z be an arbitrary set and let $z_0 \in Z \notin \mathrm{im}(f)$. Define $\beta', \beta'' : B \to Z$ by the following rules

$$\beta'(\mathrm{im}(f)) = \beta''(\mathrm{im}(f)) \qquad \beta'(z_0) \neq \beta''(z_0).$$

Then $\beta' \circ f = \beta'' \circ f$ since they agree on f's image values, but $\beta' \neq \beta''$. Thus f is not epimorphic, completing a contrapositive proof. ❦

A monomorphism (resp. epimorphism) is denoted by $\hookrightarrow$ (resp. $\twoheadrightarrow$).

[1.4.5] Remark (*Product Decomposition*)

Observe the decomposition of the Cartesian product of sets $A \times B$ in terms of inclusions (monomorphisms) and projections (epimorphisms) as follows.

$$\begin{tikzcd} A \arrow[dr, hook, "i_b"] & & A \\ & A \times B \arrow[ur, two heads, "\pi_A"] \arrow[dr, two heads, "\pi_B"] & \\ B \arrow[ur, hook, "i_a"'] & & B \end{tikzcd}$$

The inclusion maps i_b (resp. i_a) map a set to the product by $i_b(a) = (a, b)$ (resp. $i_a(b) = (a, b)$)—they are obviously monomorphic. Moreover, the projection maps π_A (resp. π_B) project a product onto a set by $\pi_A((a, b)) = a$ (resp. $\pi_B((a, b)) = b$)—they are obviously epimorphic.

We now reach the canonical decomposition, an extremely important decomposition that yields a natural isomorphism of sets (and perhaps other structures).

[1.4.6] Remark (*Canonical Decomposition*)

Let $f : A \to B$ be a map of sets from A to B. Define $\sim$ be an equivalence relation on S such that

$$a \sim b \iff f(a) = f(b).$$

Note that the map sending an element $a \in A$ to its equivalence class $[a]_\sim$ is epimorphic—we denote it $\pi : A \twoheadrightarrow A/\sim$. Note f decomposes as follows.

$$\begin{tikzcd} A \arrow[r, two heads, "\pi"'] \arrow[rrr, bend left, "f"] & A/\sim \arrow[r, "\tilde{f}"'] & \mathrm{im}(f) \arrow[r, hook, "i"'] & B \end{tikzcd}$$

In words, the map $a \mapsto f(a)$ can be decomposed canonically as follows:

(1) Take each $a \in A$ to its equivalence class $[a]_\sim \in A/\sim$. Note the canonical projection $\pi : A \twoheadrightarrow A/\sim$ is an epimorphism.

(2) Take each equivalence class $[a]_\sim \in A/\sim$ to the image of any element of the equivalence class $f(a) \in \mathrm{im}(f)$. This is justified by all elements of $[a]_\sim$ mapping to the same element in $\mathrm{im}(f)$. This is an isomorphism of sets $\tilde{f} : A/\sim \xrightarrow{\sim} \mathrm{im}(f)$.

(3) Finally, embed the elements of $\mathrm{im}(f)$ into B by the inclusion map $i : \mathrm{im}(f) \hookrightarrow B$.

The lengthy description of this process effectively serves as proof.

1.5 Categories

[1.5.1] Definition (*Category*)

A **category** C consists of:

(1) A class Obj(C) of objects of the category.

(2) For every $A, B \in \mathrm{Obj(C)}$, a set $\mathrm{Hom_C}(A, B)$ of morphisms between objects, with the following properties:

   (a) **Identity.** For every $A \in \mathrm{Obj(C)}$, there exists a morphism $1_A \in \mathrm{Hom_C}(A, A)$, called the identity on A.

   (b) **Composition.** For any $A, B, C \in \mathrm{Obj(C)}$, there is a composition operation: morphisms $f \in \mathrm{Hom_C}(A, B)$ and $g \in \mathrm{Hom_C}(B, C)$ determine a morphism $gf \in \mathrm{Hom_C}(A, C)$.

   (c) **Associativity.** For any $A, B, C, D \in \mathrm{Obj(C)}$, morphisms $f \in \mathrm{Hom_C}(A, B)$, $g \in \mathrm{Hom_C}(B, C)$, $h \in \mathrm{Hom_C}(C, D)$ compose associatively. That is, $(hg)f = h(gf)$.

   (d) **Unitality.** For any $A, B \in \mathrm{Obj(C)}$ and any morphism $f \in \mathrm{Hom_C}(A, B)$, we have that $f1_A = f$ and $1_B f = f$.

[1.5.2] Example (*Category* Set)

We can define a category Set, whose objects are all possible sets and whose morphisms are functions mapping one set to another.

From here, we provide examples of common categories.

[1.5.3] Example (*Category Induced by an Relation on a Set*)

Suppose S is a set with an relation $\sim$ that is reflexive and transitive. This determines a category whose objects are the elements of S and whose morphisms are such that whenever $a, b \in S$, $(a, b) \in \mathrm{Hom}(a, b)$ if $a \sim b$ and $\mathrm{Hom}(a, b) = \emptyset$ otherwise. We verify the axioms as follows:

(1) **Identity.** Note, for any $a \in A$, $a \sim a$ and so $1_a = (a, a) \in \mathrm{Hom}(a, a)$.

(2) **Composition.** Suppose $f \in \mathrm{Hom}(a, b)$ and $g \in \mathrm{Hom}(b, c)$ for some $a, b, c \in S$. Then note $a \sim b$ and $b \sim c$, so by transitivity, $a \sim c$, meaning the morphisms f, g determine the morphism $gf \in \mathrm{Hom}(a, c)$.

(3) **Associativity.** This follows from the transitivity of $\sim$.

(4) **Unitality.** This follows trivially from how composition is defined.

Many important categories fall out of this exam. For a trivial one, consider any set S and $=$ as its relation (which is actually an equivalence relation!). As such, all the morphisms are identity morphisms—such a category is called **discrete**.

[1.5.4] Example (*Category Induced by a Partially-Ordered Set*)

Branching off from previous discussion, we can endow $\mathbb{Z}$ with the $\le$ relation ($a \sim b$ iff. $a \le b$). This induces a category whose objects are integers and whose morphisms are connections $n \to m$ for every $n \le m$.

[1.5.5] Example (*Slice Category*)

We can create more abstract—and consequently more powerful—categories: one such example is the **slice category**. Let C be any category with $A \in \mathrm{Obj}(\mathsf{C})$. We can define a category $\mathsf{C/A}$ as follows:

(1) **Objects.** An object $f \in \mathrm{Obj}(\mathsf{C/A})$ is a morphism $f \in \mathrm{Hom}_{\mathsf{C}}(B, A)$ for any $B \in \mathrm{Obj}(\mathsf{C})$. That is, the slice category $\mathsf{C/A}$ comprises all the morphisms going to A for its objects.

(2) **Morphisms.** Suppose $f_1, f_2 \in \mathrm{Obj}(\mathsf{C/A})$ such that $f_1 \in \mathrm{Hom}_{\mathsf{C}}(C_1, A)$ and $f_2 \in \mathrm{Hom}_{\mathsf{C}}(C_2, A)$ for $C_1, C_2 \in \mathrm{Obj}(\mathsf{C})$. Then a morphism $\sigma \in \mathrm{Hom}_{\mathsf{C/A}}(f_1, f_2)$ is defined such that $f_1 = f_2\sigma$. Graphically, σ is a morphism in C making the following triangle commute.

$$\begin{tikzcd} C_1 \arrow[rr, "\sigma"] \arrow[dr, "f_1"'] & & C_2 \arrow[dl, "f_2"] \\ & A & \end{tikzcd}$$

Verification of the category axioms is trivially shown by pulling back to C, so they are omitted.

[1.5.6] Example (*Coslice Category*)

In a similar, yet opposite, sentiment, we define the **coslice category**. Let C be any category with $A \in \mathrm{Obj}(\mathsf{C})$. We can define a category A/C as follows:

(1) **Objects.** An object $f \in \mathrm{Obj}(A/\mathsf{C})$ is a morphism $f \in \mathrm{Hom}_{\mathsf{C}}(A, B)$ for any $B \in \mathrm{Obj}(\mathsf{C})$. That is, the coslice category A/C comprises all the morphisms that map A to some other object in the category.

(2) **Morphisms.** Suppose $f_1, f_2 \in \mathrm{Obj}(A/\mathsf{C})$ such that $f_1 \in \mathrm{Hom}_{\mathsf{C}}(A, C_1)$ and $f_2 \in \mathrm{Hom}_{\mathsf{C}}(A, C_2)$ for $C_1, C_2 \in \mathrm{Obj}(\mathsf{C})$. Then a morphism $\sigma \in \mathrm{Hom}_{A/\mathsf{C}}(f_1, f_2)$ is defined such that $\sigma f_1 = f_2$. Graphically, σ is a morphism in C making the following triangle commute.

$$\begin{tikzcd} C_1 \arrow[rr, "\sigma"] & & C_2 \\ & A \arrow[ul, "f_1"] \arrow[ur, "f_2"'] & \end{tikzcd} \quad = \quad \begin{tikzcd} & A \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & \\ C_1 \arrow[rr, "\sigma"] & & C_2 \end{tikzcd}$$

Note the similarities between the coslice category and the slice category.

[1.5.7] Example (*Slice Category of* $(\mathbb{Z}, \leq)$)

Consider the set $\mathbb{Z}$ endowed with the relation $\leq$, and let C be the category induced by this relation. Fix $3 \in \mathbb{Z}$, and consider the slice category $\mathsf{C}/3$. Then the objects $f \in \mathrm{Obj}(\mathsf{C}/3)$ are morphisms $f \in \mathrm{Hom}_{\mathsf{C}}(n, 3)$ for any $n \leq 3$. The morphisms are maps between two objects, so they take the form $\sigma \in \mathrm{Hom}_{\mathsf{C}}(n, m)$ for $m \leq n \leq 3$.

[1.5.8] Example (*Coslice Category* Set^*)

Let $\mathsf{C} = \mathsf{Set}$ and let A be a fixed singleton $\{*\}$. Then the objects of A/C are precisely set-functions of form $f : \{*\} \to S$. Suppose $f_1 : \{*\} \to S$ and $f_2 : \{*\} \to T$. Then a morphism $\sigma : S \to T$ is determined in the sense that $\sigma(f_1(*)) = f_2(*)$.

1.6 Morphisms

We now redefine certain concepts about morphisms categorically.

[1.6.1] Definition (*Isomorphism*)

Let C be a category and let $f \in \mathrm{Hom}_{\mathsf{C}}(A, B)$ be a morphism with $A, B \in \mathrm{Obj}(\mathsf{C})$. f is an **isomorphism** if there exists some $g \in \mathrm{Hom}_{\mathsf{C}}(b, a)$ such that $gf = 1_A$ and $fg = 1_B$.

[1.6.2] Example (*Bijections in* Set *are Isomorphisms*)

Note that morphisms in Set are precisely set-functions, and bijective set-functions are precisely the ones that have two-sided inverses, so they are isomorphisms in the Set category.

[1.6.3] Theorem (*Properties of Isomorphisms*)

(1) Isomorphisms have unique inverses.

(2) Each identity 1_A is an isomorphism and is involutory.

(3) If f is an isomorphism, then so is f^{-1}. Furthermore, $(f^{-1})^{-1} = f$.

(4) If $f \in \mathrm{Hom}_{\mathsf{C}}(A, B)$ and $g \in \mathrm{Hom}_{\mathsf{C}}(B, C)$ are isomorphisms, then $gf \in \mathrm{Hom}_{\mathsf{C}}(A, C)$ is an isomorphism whose inverse is given by $(gf)^{-1} = f^{-1}g^{-1}$.

Proof. These are straight forward arguments whose proofs generalize cleanly from naive set theory, so we omit the proof.❦

[1.6.4] Definition (*Endomorphism, Automorphism*)

Let C be a category. A morphism f in C is an **endomorphism** if it maps an object A to itself. The set of all endomorphisms is denoted by $\mathrm{End}_{\mathsf{C}}(A)$. f is an **automorphism** if it is an endomorphism that is an isomorphism. The set of all automorphisms is denoted by $\mathrm{Aut}_{\mathsf{C}}(A) \subseteq \mathrm{End}_{\mathsf{C}}(A)$.

[1.6.5] Definition (*Groupoid*)

A category C is a **groupoid** if every morphism it comprises is an isomorphism.

Monomorphisms and epimorphisms were already defined on Set, but note their equivalence to injectivity and surjectivity—as established in Set—does not necessarily hold in other categories.

1.7 Exercises

[1.7.1] PROBLEM

Let $A, B \in \mathrm{Obj}(\mathsf{C})$ with $f \in \mathrm{Hom}_{\mathsf{C}}(A, B)$.

(1) Show that if f has a right inverse $g \in \mathrm{Hom}_{\mathsf{C}}(B, A)$, then f is an epimorphism.

(2) Show that the converse is not true by explicitly constructing a category and an epimorphism without a right-inverse.

Proof. Let $\beta', \beta'' \in \mathrm{Hom}_{\mathsf{C}}(C, A)$, with $C \in \mathrm{Obj}(\mathsf{C})$, be such that

$$\beta' \circ f = \beta'' \circ f.$$

Apply g to both sides, yielding

$$\beta' \circ (f \circ g) = \beta'' \circ (f \circ g) \implies \beta' \circ 1_B = \beta'' \circ 1_B \implies \beta' = \beta'',$$

completing (1).

To do (2), consider the category C induced by the partial ordering $\leq$ on $\mathbb{Z}$. Then fix the morphism $f : 2 \twoheadrightarrow 3$, which is right-cancellative by inspection (and thus epic). However note f has no right inverse (3 can never map to 2), completing the proof. ❦

1.8 Universal Properties

The idea behind universal properties is that they generalize specific constructions by emulating their behavior.

Consider $\varnothing$ in the category Set. Note that, for any set $S \in \text{Obj}(\text{Set})$, there is precisely one set-function/morphism $f : \varnothing \to S$ (the empty function). There is no choice to be made for how to map $\varnothing$ to S, obviously.

Conversely, consider singletons of the form $\{*\}$ in the category Set. Note that, for any set $S \in \text{Obj}(\text{Set})$, there is precisely one set-function $f : S \to \{*\}$ (the constant function $f \equiv *$). There is no choice to be made for how to map S to $\{*\}$, obviously.

We generalize this behavior to more general categories using the notion of initial and terminal objects.

[1.8.1] Definition (*Initial, Terminal Objects*)

An object $I \in \text{Obj}(\mathsf{C})$ is said to be **initial** if, for any $A \in \text{Obj}(\mathsf{C})$, there is precisely one morphism that maps I to A. Conversely, an object $T \in \text{Obj}(\mathsf{C})$ is said to be **terminal** if, for any $A \in \text{Obj}(\mathsf{C})$, there is precisely one morphism that maps A to I. That is,

$$\forall I \in \text{Obj}(\mathsf{C}) : \qquad \text{Hom}_{\mathsf{C}}(I, A) \text{ is a singleton,}$$
$$\forall T \in \text{Obj}(\mathsf{C}) : \qquad \text{Hom}_{\mathsf{C}}(A, T) \text{ is a singleton.}$$

[1.8.2] Remark (*Initial and Terminal Objects of* Set)

Note then that $\varnothing$ is the initial object of Set and that all singletons $\{*\}$ are terminal objects in Set. But note that $\varnothing$ is the singular, unique initial object of C, while there are infinitely many singletons $\{*\}$ that are terminal objects of C. However, we will show that all initial and terminal objects are isomorphic to each other.

[1.8.3] Theorem (*Initial and Terminal Objects Are Unique up to Isomorphism*)

Let C be a category.

(1) If $I_1, I_2 \in \text{Obj}(\mathsf{C})$ are initial, then $I_1 \cong I_2$.

(2) If $T_1, T_2 \in \text{Obj}(\mathsf{C})$ are terminal, then $T_1 \cong T_2$.

Proof. To prove (1), suppose I_1 and I_2 are initial objects of C. Then note the morphisms $f : I_1 \to I_2$ and $g : I_2 \to I_1$ are unique by the property of I_1, I_2 being initial. Note then that $gf \in \text{End}_{\mathsf{C}}(I_1)$ and $fg \in \text{End}_{\mathsf{C}}(I_2)$. But note that morphisms of form $I_1 \to I_1$ and $I_2 \to I_2$ are unique since the two objects are initial, so $|\text{End}_{\mathsf{C}}(I_1)| = |\text{End}_{\mathsf{C}}(I_2)| = 1$. Moreover, these sets must contain an identity morphism, so in reality fg and gf are forced to be identity morphisms. Since f has a two-sided inverse g, it is an isomorphism, completing the proof for (1). The argument for (2) follows in an entirely symmetric way. ❦

A construction satisfies a **universal property** if, loosely speaking, it is the initial or terminal object of some category. We describe this by providing intuition for certain constructions satisfying universal properties.

[1.8.4] Remark (*Quotient of Set Is Universal*)

Suppose A is some set endowed with an equivalence relation $\sim$. Consider the following statement:

"$A/\sim$ is universal with respect to the property that A maps to any set Z in such a way that elements equivalent under $\sim$ have the same image."

To decipher this, we systematically construct an accessory category where this universal construction manifests as an initial or terminal object. In general, whenever a construction is the solution to a universal problem, its accessory category is built as follows:

- **Objects:** The possible "setups" satisfying the described property, generally a set paired with constraining morphisms.
- **Morphisms:** Standard functions between the sets that make the diagrams formed by the constraining morphisms commute.
- **Initial/Terminal Objects:** The universal solution is precisely the initial or terminal object of this category (unique up to isomorphism).

We now design this accessory category for our specific quotient property.

- **Objects:** Pairs (φ, Z) where Z is a set and $\varphi : A \to Z$ is a function satisfying $\varphi(a') = \varphi(a'')$ whenever $a' \sim a''$.
- **Morphisms:** A morphism from (φ_1, Z_1) to (φ_2, Z_2) is defined as a function $\sigma : Z_1 \to Z_2$ that respects the mappings from A. That is, the following diagram must commute:

$$\begin{tikzcd} Z_1 \arrow[rr, "\sigma"] & & Z_2 \\ & A \arrow[ul, "\varphi_1"] \arrow[ur, "\varphi_2"'] & \end{tikzcd}$$

Fix some arbitrary object (φ, Z). An initial object takes the form $(\overline{\varphi}, I)$ such that there exists a *unique* morphism $\sigma : (\overline{\varphi}, I) \to (\varphi, Z)$. Consider the canonical projection $\pi : A \to A/\sim$. Since $a' \sim a'' \implies \pi(a') = \pi(a'')$, the pair $(\pi, A/\sim)$ is a valid object. To satisfy the commutative diagram $(\sigma \circ \pi = \varphi)$, we are forced to define $\sigma([a]) = \varphi(a)$. Because φ respects $\sim$, this σ is uniquely determined and well-defined. Thus, $(\pi, A/\sim)$ is indeed initial.

Conversely, a terminal object takes the form $(\overline{\varphi}, T)$ such that there exists a *unique* morphism $\sigma : (\varphi, Z) \to (\overline{\varphi}, T)$. It makes sense to conjecture that $\overline{\varphi}$ sends every element of A to a constant $*$, making $T = \{*\}$. Since $\mathrm{im}(\overline{\varphi})$ is a singleton, all equivalent elements of A trivially have the same image, satisfying the object constraint. Moreover, $\sigma : Z \to T$ is unique since there is only one possible function mapping any set Z to a singleton. This unique σ trivially satisfies the required commutative diagram $(\sigma \circ \varphi = \overline{\varphi})$, proving $(\overline{\varphi}, T)$ is indeed terminal.

[1.8.5] Remark (*Product of Sets is Universal*)

Fix two sets A, B. We seek to show the product $A \times B$ satisfies a universal property. Consider the **projections** of $A \times B$ onto A and B.

$$\begin{tikzcd}
& A \times B \arrow[dl, "\pi_A"'] \arrow[dr, "\pi_B"] & \\
A & & B
\end{tikzcd}$$

We observe that the set in question is $A \times B$, constrained by morphisms π_A and π_B. Let's consider an auxiliary category where sets of the form (X, f_A, f_B) are objects—thus $(A \times B, \pi_A, \pi_B)$ form objects, and we have the following diagram.

$$\begin{tikzcd}
A & & \\
& A \times B \arrow[ul, "\pi_A"'] \arrow[dl, "\pi_B"] & X \arrow[l, "\sigma"'] \arrow[ull, bend right, "f_A"'] \arrow[dll, bend left, "f_B"] \\
B & &
\end{tikzcd}$$

$\sigma : X \to A \times B$ must be oriented in this direction to make the diagram commute. Now we simply show $A \times B$ is terminal.

> **Proof.** Fix any $x \in X$. Then note $(\pi_A \circ \sigma)(x) = f_A(x)$ and $(\pi_B \circ \sigma)(x) = f_B(x)$ by commutativity of the diagram. Suppose any other function, say σ', was such that $\sigma' : X \to A \times B$ and made the diagram commute. But then, for each $x \in X$, the image under σ and σ' would be equivalent, and thus $\sigma \equiv \sigma'$, completing the uniqueness proof.

We sum this up by saying the product of two sets $A \times B$ is universal with respect to the two projection maps onto A and B.

[1.8.6] Remark (*Disjoint Union of Sets is a Coproduct*)

The coproduct is the dual of the product, where dualization acts by flipping all the arrows in a category. It is trivial to show the disjoint union in Set is an example of the coproduct.

Previously, we defined the Cartesian product and disjoint union by set-theoretic means and ran into the problem of these definitions to identify unique sets *up to isomorphism*. With category theory, we have defined these concepts not as sets, but by how they relate to their constituents and how they solve a universal problem. This is a perfectly valid definition that does not have to resolve this ambiguity.