2 Complex Functions

2.1 Limits and Continuity

[2.1.1] Definition (*Limit*)

A function $f : A \to B$ is said to have a **limit** at some $a \in \mathrm{Lim}(A)$ (denoted $\lim_{x\to a} f(x) = A$) if, for every $\varepsilon > 0$, there is some $\delta > 0$ such that

$$|x - a| < \delta \implies |f(x) - f(a)| < \varepsilon.$$

Immediately from the definition, we see that for a function whose domain is an open subset of the complex numbers, it follows that

$$\lim_{x\to a} f(x) = A \implies \lim_{x\to a} \overline{f(x)} = \overline{A}.$$

The usual notion of continuity is lifted from real analysis.

[2.1.2] Definition (*Derivative*)

A function $f : A \to B$ is said to be **differentiable** at some $a \in \mathrm{Lim}(A)$ if

$$\lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

exists. Then $f'(a)$ takes the value of this limit.

[2.1.3] Example (*Derivatives of Functions*)

There are four classes of functions to consider in the context of complex analysis.

(1) **Real-to-Real.** For a function $f : \mathbb{R} \to \mathbb{R}$, the theory of differentiation is lifted from real analysis.

(2) **Real-to-Complex.** Let $f : \mathbb{R} \to \mathbb{C}$. Indeed, we see that $f(t) = x(t) + iy(t)$, and from multivariate analysis, we recall $f'(t) = x'(t) + iy'(t)$. There is nothing new here.

(3) **Complex-to-Real.** Let $f : \mathbb{C} \to \mathbb{R}$. From multivariate analysis, for a limit to exist, a limit through all paths must exist and be equivalent. If f is differentiable at some $a \in \mathbb{C}$, note that approaching the difference quotient from the real axis produces a purely real number, whereas approaching the difference quotient from the imaginary axis produces a purely imaginary number. Thus $f'(a) = 0$, and so $f' \equiv 0$ at points it is defined on.

(4) **Complex-to-Complex.** This is what we devote our time in complex differentiation towards. It is far less trivial than the other cases.

2.2 Holomorphic Functions

[2.2.1] Definition (*Holomorphic Function*)

We say a function $f : A \to \mathbb{C}$, where $A \subseteq \mathbb{C}$, is **differentiable** at some $z \in \mathrm{Lim}(A)$ if

$$f'(z) = \lim_{h\to 0} \frac{f(z+h) - f(z)}{h}$$

is defined. If f' is defined at all interior points of A, we say f is **holomorphic**.

[2.2.2] Corollary (*Holomorphicity implies Continuity*)

If $f : A \to \mathbb{C}$, where $A \subseteq \mathbb{C}$, is complex differentiable at some $z \in A$, then f is continuous at z.

> **Proof.** Note that $\lim_{h\to 0} f(z+h) - f(z) = \lim_{h\to 0} h \times \dfrac{f(z+h) - f(z)}{h} = 0 \times f'(z) = 0$ from standard multivariate limit theory. ❦

[2.2.3] Remark (*Derivation of Cauchy-Riemann Equations*)

Recall that for a limit to exist in $\mathbb{C} \cong \mathbb{R}^2$, limits along all paths must exist and be equal. Treating $f : A \to \mathbb{C}$, where $A \subseteq \mathbb{C}$, from this multivariate lens, we write $f(z) = u(z) + i\,v(z)$, as a vector function. For simplicity, let's say that $\mathrm{Re}(z) = x$ and $\mathrm{Im}(z) = y$.

Leaving y fixed and allowing x to vary (traveling along $\mathbb{R}$), we have that

$$f'(z) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}.$$

Leaving x fixed and allowing y to vary (traveling along $i\mathbb{R}$), and letting our step $h = ik$ for $k \in \mathbb{R}$, we have that

$$f'(z) = \lim_{k\to 0} \frac{f(z+ik) - f(z)}{ik} = -i\left(\frac{\partial f}{\partial y}\right) = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}.$$

Letting these equal eachother, we arrive at the **Cauchy-Riemann equations**:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

The Cauchy-Riemann equations have major implications for holomorphic functions. For starters, $f'(z)$ can be expressed in four distinct ways in terms of the partials of u, v. Additionally, we can write $|f'(z)|$ in a number of ways, with a very striking expression being

$$|f'(z)|^2 = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} = \frac{\partial(u,v)}{\partial(x,y)}.$$

Thus the squared norm of the derivative of a holomorphic function is exactly its Jacobian when written out as a vector function. This has many geometric implications, leading to the interpretation of holomorphic functions as conformal maps, which we will visit later.

Eventually, we will show that being once complex-differentiable is the same as being infinitely complex-differentiable, and this will imply Clairaut's Theorem (that mixed partials are equal under nice conditions). From this and the Cauchy-Riemann equations, we obtain

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

$$\Delta v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

A function u that satisfies Laplace's equation $\Delta u = 0$ is said to be **harmonic**. In the context of Cauchy-Riemann equations, the function v is said to be **conjugate harmonic** to u.

[2.2.4] Theorem (*Criterion for Holomorphicity*)

A function $f(z) = u(z) + iv(z)$ is **holomorphic** if and only if u, v have continuous, first-order partial derivatives that satisfy the Cauchy-Riemann equations.

Proof.

$(\Longrightarrow)$: This is implied from taking the limit of the difference quotient along $\mathbb{R}$ and $i\mathbb{R}$.

$(\Longleftarrow)$: Suppose u, v have continuous first-order partial derivatives that satisfy the Cauchy-Riemann equations. From multivariate theory, note that

$$u(x+h, y+k) - u(x, y) = u_x h + u_y k + o_1(h+ik),$$
$$v(x+h, y+k) - v(x, y) = v_x h + v_y k + o_2(h+ik).$$

Then note

$$f(z+[h+ik]) - f(z) = (u_x + iv_x)(h+ik) + [o_1(h+ik) + io_2(h+ik)],$$

meaning

$$f'(z) = \lim_{h+ik\to 0} \frac{f(z+[h+ik]) - f(z)}{h+ik} = u_x + iv_x.$$

❦

2.3 Polynomials

Note that all polynomials are holomorphic. For now, we will accept the Fundamental Theorem of Algebra without proof and note that any polynomial p over $\mathbb{C}$ takes the unique form

$$p(z) = c(x - a_1)\cdots(x - a_n),$$

for not necessarily distinct $a_1, \ldots, a_n$. We say each a_k is a **zero** of p of order h, where h is the number of times a_k coincides with the other roots.

Note that the order of a root can be determined by calculus. If ζ is a root of $p(z)$ of order n, then note

$$p(\zeta) = p'(\zeta) = \cdots = p^{(n-1)}(\zeta) = 0,$$

whereas $p^{(n)}(\zeta) \neq 0$. In other words, the order of ζ is n if and only if $p^{(n)}(\zeta)$ is the first nonvanishing derivative at ζ.

[2.3.1] Theorem (*Gauss-Lucas Theorem*)

Suppose a convex hull is defined by $p(z)$'s roots in the complex plane. Then $p'(z)$ has all its roots contained in the convex hull.

Proof. Suppose z_0 is a root of p'. If z_0 is a root of p, then we are done—otherwise, note $p(z_0) \neq 0$. Then

$$0 = \frac{p'(z_0)}{p(z_0)} = \sum_{k=1}^{n} \frac{1}{z_0 - \lambda_k} = \sum_{k=1}^{n} \frac{z_0 - \lambda_k}{|z_0 - \lambda_k|^2} = \sum_{k=1}^{n} m_k(z_0 - \lambda_k),$$

where $\lambda_1, \ldots, \lambda_n$ are the roots of p and $m_k = 1/|z_0 - \lambda_k|^2 \in \mathbb{R}$. Rewriting yields

$$z_0 = \frac{\sum_k m_k \lambda_k}{\sum_k \lambda_k},$$

a weighted linear combination of the roots of p (the points that define the hull), completing the proof. ❦

2.4 Rational Functions

[2.4.1] Definition (*Rational Function, Poles*)

A **rational function** $R(z)$ is the quotient of two polynomials:

$$R(z) = \frac{P(z)}{Q(z)}.$$

For simplicity, suppose $R(z)$ is simplified fully, and so P and Q share no common factors. We will say $R(z) = \infty$ at the zeros of $Q(z)$, where $\infty \in \hat{\mathbb{C}}$. The zeros at $Q(z)$ are referred to as the **poles** of $R(z)$, and the **order** of a pole is defined to be the order/multiplicity of the root of $Q(z)$.

[2.4.2] Remark (*Differentiation Increments Order of Pole*)

Note that, for $R(z) = P(z)/Q(z)$, we have that

$$R'(z) = \frac{P'(z)Q(z) - P(z)Q'(z)}{[Q(z)]^2}.$$

Note that if z_0 is a pole of R, then z_0 is also a pole for R'—we seek to understand how differentiation affects the order of a pole. Write $Q(z) = (z - z_0)^n A(z)$, where n is the order of $R(z_0)$. Then note

$$\begin{aligned} R'(z) &= \frac{P'(z)(z-z_0)^n A(z) - nP(z)(z-z_0)^{n-1}A'(z)}{[Q(z)]^2} \\ &= \frac{(z-z_0)^{n-1}[P'(z)(z-z_0)A(z) - nP(z)A'(z)]}{(z-z_0)^{2n}A^2(z)} \\ &= \frac{V(z)}{(z-z_0)^{n+1}A^2(z)}, \end{aligned}$$

where $V(z)$ is the residual polynomial such that $V(z_0) \neq 0$. Thus $R'(z_0)$ has order $n+1$.

[2.4.3] Remark (*Poles at Infinity*)

We note the value of a rational function at a pole is at $\infty \in \hat{\mathbb{C}}$. But what about the behavior of a rational function at ∞? That is, how do we determine if a rational function $R(z)$ has a pole at ∞?

We can let $z \to \infty$ and observe $R(z)$, but this wouldn't determine the order of a pole at ∞. Instead, we define $R_1(z) = R(1/z)$, and observe what happens to R_1 as $z \to 0$. This is perfect to determine the behavior of R at ∞.

For some arbitrary rational function, note

$$R(z) = \frac{a_0 + a_1 z + \cdots + a_n z^n}{b_0 + b_1 z + \cdots + b_m z^m} \qquad R_1(z) = z^{m-n}\frac{a_0 z^n + a_1 z^{n-1} + \cdots + a_n}{b_0 z^n + b_1 z^{n-1} + \cdots + b_n}.$$

Accordingly, if $m > n$, then $R(\infty) = 0$ with order $m - n$. If $m < n$, then $R(\infty)$ has a pole of order $n - m$. If $m = n$, then $R(\infty) = a_n / b_m$.

[2.4.4] Remark (*Number of Poles and Zeros Coincide*)

Again, write $R(z) = P(z)/Q(z)$ in its presentation as in the last remark, where $\deg(P) = n$ and $\deg(Q) = m$. In the finite plane $\mathbb{C}$, we note R has n zeros and m poles.

Now let's consider this in the one-point compactification of the plane, $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. If $m > n$, we have a zero at ∞ of order $m - n$; if $m < n$, we have a pole of order $n - m$; if $m = n$, we have neither. In any case, it follows that the number of zeros and the number of poles of R, as considered in the infinite plane, coincide and are equivalent to $\max(m, n)$. We say that the **order** of a rational function is the number of zeros/poles it has in the infinite plane.

Note that if $R(z)$ has order p, then $R(z) - a$ obviously also has order p.

[2.4.5] Remark (*Partial Fraction Decomposition in the Infinite Plane*)

We seek to show that any rational function $R(z) = P(z)/Q(z)$ has a partial fraction decomposition. Partial fraction decomposition is a method to encode the behavior of R at its poles.

To start, suppose R has a pole at ∞. Then $\deg(P) > \deg(Q)$, and so by long division, we have that

$$R(z) = \tilde{G}(z) + \frac{S(z)}{Q(z)} = G(z) + \left(c_0 + \frac{S(z)}{Q(z)}\right) = G(z) + H(z),$$

where $c_0 = G(0)$. What we have done is split $R(z)$ as the sum of two parts: one part that blows up at ∞ (this encodes the behavior of R at ∞), and one part that is finite at ∞. After normalizing G so that $G(0) = 0$, we say G is the singular part of $R(z)$ at ∞, whereas H is the finite part at ∞. This terminology applies to other poles, as we will soon see.

Now let $\lambda_1, \dots, \lambda_n$ denote the distinct, finite poles of R. To create a decomposition similar to the one at ∞, we use a transformation that sends λ_k to ∞. $\zeta = (z - \lambda_k)^{-1}$ is the appropriate transformation, since when writing $z = \lambda_k + \zeta^{-1}$, the point λ_k corresponds to a pole at ∞. We now apply this type of decomposition

$$R\left(\lambda_k + \frac{1}{\zeta}\right) = G_k(\zeta) + H_k(\zeta) \implies R(z) = G_k\left(\lambda_k + \frac{1}{\zeta}\right) + H_k\left(\lambda_k + \frac{1}{\zeta}\right),$$

where G_k is the singular part of $R(z)$ at λ_k and H_k is the finite part.

Finally, we seek to show

$$R(z) = G(z) + \sum_{k=1}^{n} G_k\left(\frac{1}{z - \lambda_k}\right).$$

Define

$$f(z) = R(z) - G(z) - \sum_{k=1}^{n} G_k\left(\frac{1}{z - \lambda_k}\right).$$

We will first show f has no poles. We notice immediately the only candidates at which f has a pole are $\lambda_1, \dots, \lambda_n, \infty$.

We first observe how f behaves at each λ_k. First, write

$$f(z) = \left[R(z) - G_k\left(\frac{1}{z - \lambda_k}\right)\right] - G(z) - \sum_{k=1, j \neq k}^{n} G_j\left(\frac{1}{z - \lambda_k}\right).$$

Note the first term reduces to $H_k((z - \lambda_k)^{-1})$, which is finite at the pole, and $G(z)$ and the sum do not diverge at λ_k. Thus f does not have a pole at λ_k. Now, observe how f behaves at ∞. Note $\lim_{z \to \infty} R(z) - G(z) = \lim_{z \to \infty} H(z)$, which is finite, and the other terms are also finite as $z \to \infty$. Thus f has no pole at ∞ and thus has no poles at all.

The only rational functions that have no poles are identically constant. Absorbing this constant into $G(z)$, we have proven

$$R(z) = G(z) + \sum_{k=1}^{n} G_k\left(\frac{1}{z - \lambda_k}\right).$$