18.100B - Real Analysis

September 10th, 2026

[0.0.1] Axiom (*Field Axioms*)

A field $\mathbb{F}$ is a set together with two operations, addition and multiplication as depicted below, that obeys a list of axioms

$$+ : \mathbb{F} \times \mathbb{F} \to \mathbb{F} \quad \cdot : \mathbb{F} \times \mathbb{F} \to \mathbb{F}.$$

(1) **Commutativity.** $x + y = y + x$ and $xy = yx$.

(2) **Associativity.** $(x + y) + z = x + (y + z)$ and $(xy)z = x(yz)$.

(3) **Distributivity.** $x(y + z) = xy + yz$.

(4) **Identity.** There is some $0 \in \mathbb{F}$ (resp. $1 \in \mathbb{F}$) such that $a + 0 = a$ (resp. $a \cdot 1 = a$) for every $a \in \mathbb{F}$.

(5) **Inverse.** For every $a \in \mathbb{F}$, there is some unique element $-a \in \mathbb{F}$ (resp. $a^{-1} \in \mathbb{F}$) such that $a + (-a) = 0$ (resp. $aa^{-1} = 1$). In the case of multiplication, we do not consider 0 when checking for inverse.

(6) **Nondegeneracy.** $1 \neq 0$.

The field operations, by way of definition, automatically give rise to **closure**: that, for any $x, y \in \mathbb{F}$, we have that $x + y \in \mathbb{F}$ and $x \cdot y \in \mathbb{F}$.

[0.0.2] Axiom (*Order Axioms*)

Let $\mathbb{F}$ be a field. For any $a, b \in \mathbb{F}$, we say $a = b$ if they are the same element in the set. We may also introduce a symbol $<$ for comparison. A field $\mathbb{F}$ is said to be totally ordered if it obeys the following axioms.

(1) **Trichotomy.** For any $x, y \in \mathbb{F}$, one and only one of these statements hold: $x = y$, $x > y$, $x < y$.

(2) **Transitivity.** If $x < y$ and $y < z$, then $x < z$.

(3) **Monotonicity.** If $x < y$, then $x + z < y + z$ for any $z \in \mathbb{F}$. Also, $xz < yz$ if $z > 0$.

[0.0.3] Axiom (*Upper Bound, Dedekind Completeness*)

Suppose $\mathbb{F}$ is an ordered field. If $S \subseteq \mathbb{F}$, we say b is an **upper bound** for S if $x \leq b$ for every $x \in S$. We say that b is a **least upper bound** for S if, of all upper bounds on S, b is less than or equal to all other upper bounds. A field $\mathbb{F}$ is said to be **Dedekindcomplete** if every subset of $\mathbb{F}$ that is nonempty and bounded above has a least upper bound.

[0.0.4] Definition (*Definition of the Set of Reals*)

$\mathbb{R}$ is the unique ordered and Dedekind complete field.

Note that Dedekind completeness is equivalent to Cauchy completeness and the Archimedean property.

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[0.0.5] DEFINITION (*Sequence of Reals*)

A **sequence** is a function $f : \mathbb{Z}_+ \to \mathbb{R}$. The n-th term of f is written as $x_n = f(n)$. The entire sequence may be denoted as a list $(x_1, x_2, x_3, \dots)$, or $(x_n)_{n=1}^{\infty}$.

[0.0.6] DEFINITION (*Limit of a Sequence*)

Let $(x_n)_{n=1}^{\infty} \subseteq \mathbb{R}$ be a sequence of reals. We say that the **limit** of the sequence x_n is L if, for every $\varepsilon > 0$, there exists some $N \in \mathbb{Z}_+$ such that $|x_n - L| < \varepsilon$ for every $n \geq N$. We denote the limiting behavior in the following various forms.

$$\lim_{n\to\infty} x_n = L \quad \lim(x_n) = L \quad x_n \to L$$

[0.0.7] THEOREM (*Sum and Product of Limits*)

Let $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ be sequences of reals convering to x and y respectively.

(1) $\lim(x_n + y_n) = x + y$.

(2) $\lim(x_n y_n) = xy$.

Proof.

(1) Fix $\varepsilon > 0$. There is some $N_1, N_2 \in \mathbb{Z}_+$ such that $|x_n - x| < \varepsilon/2$ and $|y_n - y| < \varepsilon/2$ for every $n \geq N$, where $N = \max\{N_1, N_2\}$. Then note

$$|(x_n + y_n) - (x + y)| \leq |x_n - x| + |y_n - y| < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

for every $n \geq N$ as desired.

(2) Fix $\varepsilon > 0$. Since $x_n \to x$, we have that there exists some $A \in \mathbb{Z}_+$ such that $|x_n - x| < 1$ for every $n \geq A$. We are justified in choosing

$$M = \max\{|x_1|, |x_2|, \dots, |x_{A-1}|, |x| + 1\},$$

and note $|x_n| < M$ for all $n \in \mathbb{Z}_+$.

Let $N_1 \in \mathbb{Z}_+$ be such that $|x_n - x| < \varepsilon/2(|y| + 1)$ and $N_2 \in \mathbb{Z}_+$ be such that $|y_n - y| < \varepsilon/2M$. Choose $N = \max\{N_1, N_2\}$, and note

$$|x_n y_n - xy| = |x_n y_n - y x_n + y x_n - xy| \leq |x_n||y_n - y| + |y||x_n - x| < |x_n| \frac{\varepsilon}{2M} + |y| \frac{\varepsilon}{2(|y| + 1)} < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

for all $n \geq N$. Thus $x_n y_n \to xy$ as desired.

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[0.0.8] Definition (*Limit of a Function*)

Let $A \subseteq \mathbb{R}$ be some subset of $\mathbb{R}$, and consider a function $f : A \to \mathbb{R}$. For some limit point a of A, we say the **limit** of $f(x)$ as x approaches a is L if, for any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon$$

for all $x \in A$. We write this as

$$\lim_{x \to a} f(x) = L.$$

[0.0.9] Definition (*Continuity of a Function*)

Let $A \subseteq \mathbb{R}$ be some subset of $\mathbb{R}$, and consider a function $f : A \to \mathbb{R}$. For some $a \in A$, we say f is **continuous** at a if, for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|x - a| < \delta \implies |f(x) - f(a)| < \varepsilon$$

for all $x \in A$.

[0.0.10] Definition (*Derivative of a Function*)

Let $A \subseteq \mathbb{R}$ be some subset of $\mathbb{R}$, and consider a function $f : A \to \mathbb{R}$. For some interior point $a \in A$, provided the limit exists, we say the **derivative** of f at a is given by

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$